COMMENTS

Comments are short papers which criticize or correct papers of other authors previously published in the **Physical Review.** *Each Comment should state clearly to which paper it refers and must be accompanied by a brief abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.*

Comment on "Strange nonchaotic attractors in autonomous and periodically driven systems"

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The problem of the existence of strange nonchaotic attractors (SNA's) in autonomous systems is discussed. It is demonstrated that the recently reported example of a SNA in an autonomous system [V. S. Anishchenko *et al.*, Phys. Rev. E **54**, 3231 (1996)] is in fact a chaotic attractor with positive largest Lyapunov exponent. [S1063-651X(97)11810-4]

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Strange nonchaotic attractors (SNA's) typically appear in quasiperiodically forced nonlinear dynamical systems. These attractors were described by Grebogi *et al.* in 1984 [1] and since then investigated in a number of numerical [2–15] and experimental [16,17] studies. A typical system considered in most of these works is a nonlinear continuous- or discrete-time oscillator with a quasiperiodic two-frequency forcing. Strange nonchaotic attractors, which are observed in such a system, exhibit some properties of regular as well as chaotic systems. Like regular attractors they have only negative and zero (related to the quasiperiodic forcing) Lyapunov exponents; like usual chaotic attractors they have a strange geometrical structure. Also, their correlation properties lie in between order and chaos: as shown in [12,18], they can have a singular continuous spectrum. It is noteworthy that the SNA appears in studies of properties of spectra and eigenfunctions of quantum systems with a quasiperiodic potential [2,19,20]. Mathematical studies of SNA's are still in the beginning phase [21,22].

One of the main questions in this field is whether a SNA can be observed in an *autonomous* system, i.e., without explicit external quasiperiodic forcing (in mathematical language, such systems cannot be divided into a driving and a responsing parts). Grebogi *et al.* in 1984 [1] already addressed this question and studied the influence of small perturbations to the system which change the quasiperiodically forced system to an autonomous one. In all tested cases they could not find a strange nonchaotic attractor, but only periodic or chaotic attractors even if the perturbations to the system were rather small in size (about 10^{-4}). These findings suggest that SNA's cannot occur in autonomous systems. Recently, a paper appeared [23] where an apparently strange nonchaotic attractor in an autonomous four-dimensional mapping was reported. The authors calculated the largest Lyapunov exponent, and found that its values are nearly zero both below and above the transition at which a geometrical strangeness appears (Fig. 1 in [23]).

We have performed numerical experiments with the four-dimensional mapping used in [23]

$$x_{n+1}=x_n+\Omega_1-\frac{K_1}{2\pi}\sin(2\pi x_n)+\gamma_1 y_n$$

$$+A\cos(2\pi u_n) \bmod 1,$$

$$y_{n+1}=\gamma_1 y_n-\frac{K_1}{2\pi}\sin(2\pi x_n),$$

$$u_{n+1}=u_n+\Omega_2-\frac{K_2}{2\pi}\sin(2\pi u_n)+\gamma_2(y_n+v_n) \bmod 1,$$

$$v_{n+1}=\gamma_2(y_n+v_n)-\frac{K_2}{2\pi}\sin(2\pi u_n).$$

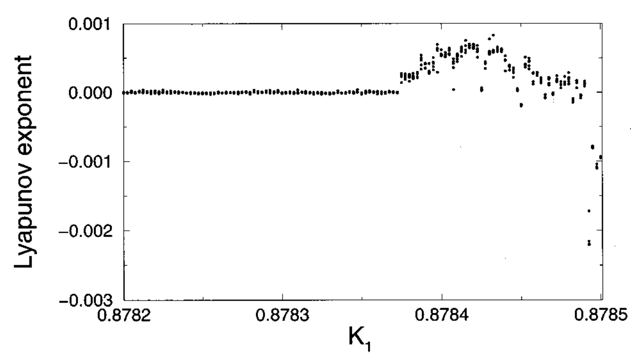

FIG. 1. The largest Lyapunov exponent vs the parameter K_1. In the numerical experiment the first 10^6 iterations have been discarded to get rid of transients; the next 10^6 iterations have been used for the calculation of the exponent.

Following [23], we fixed the parameters $\Omega_1 = 0.5$, $\Omega_2 = 0.5(\sqrt{5}-1)$, $K_2 = 0.03$, $A = 0.4$, $\gamma_1 = \gamma_2 = 0.001$ and studied the dynamics depending on K_1. Near the transition at $K_1 \approx 0.87837$ we calculated the largest Lyapunov exponent (see Fig. 1). The vertical scattering of points for each value of K_1 gives an impression on the averaging error. From Fig. 1 we conclude that the transition at $K_1 \approx 0.878\,37$ is a transition to a chaotic attractor with a positive largest Lyapunov exponent. Thus the claim of Ref. [23] on the observation of a SNA in the autonomous system above is not confirmed by accurate calculations. The existence of such attractors in autonomous systems remains an open question.

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