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Phase synchronization of chaotic oscillators by external driving

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Abstract

We extend the notion of phase locking to the case of chaotic oscillators. Different definitions of the phase are discussed, and the phase dynamics of a single self-sustained chaotic oscillator subjected to external force is investigated. We describe regimes where the amplitude of the oscillator remains chaotic and the phase is synchronized by the external force. This effect is demonstrated for periodic and noisy driving. This phase synchronization is characterized via direct calculation of the phase, as well as by implicit indications, such as the resonant growth of the discrete component in the power spectrum and the appearance of a macroscopic average field in an ensemble of driven oscillators. The Rössler and the Lorenz systems are shown to provide examples of different phase coherence properties, with different response to the external force. A relation between the phase synchronization and the properties of the Lyapunov spectrum is discussed.

Key words: Synchronization; Phase dynamics; Chaotic oscillator; Phase locking

1. Introduction

Synchronization is a basic phenomenon in physics, discovered at the begining of the modern age of science by Huygens [1]. In the classical sense, synchronization means adjustment of frequencies of periodic oscillators due to weak interaction [2–4]. This effect is well studied and finds a lot of practical applications [5].

Recently, with widespread studies of chaotic oscillations, the notion of synchronization has been generalized to the latter case. In this context, different phenomena exist which are usually refered to as "synchronization", so one needs a more precise description to specify them. Due to a strong interaction of two (or a large number of) identical chaotic systems, their states can coincide, while the dynamics in time remains chaotic [6,7]. This case can be denoted as "complete synchronization" of chaotic oscillators. It can be easily generalized to the case of slightly

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non-identical systems [7] or the interacting subsystems [8]. A different approach is based on the calculation of the attractor dimension of the whole system and its comparison with the partial dimensions calculated in the phase subspaces formed by the coordinates of each interacting oscillator [9,10]. Another well-studied effect is the "chaos-destroying" synchronization, when a periodic external force acting on a chaotic system destroys chaos and a periodic regime appears [11] (or, in the case of an irregular forcing, the driven system follows the behavior of the force [12]). This effect occurs for a relatively strong forcing as well. A characteristic feature of these phenomena is the existence of a threshold coupling value (depending on the Lyapunov exponents of individual systems) [6,7,13,14].

In another approach, an overlapping of power spectra of certain observables in the interacting systems has been studied. It has been shown that due to interaction, the widths of the peaks of the power spectra may become practically equal, and the peak's frequencies become closer [15]. Because of an analogy to the coincidence of frequencies of synchronized periodic systems this effect has been interpreted as synchronization and has been quantified by means of cross-correlation functions [15] and cross-spectra [16].

We have recently described the effect of *phase synchronization* of chaotic systems [17], which is mostly close to synchronization of periodic oscillations, where only the phase locking is important, while no restriction on the amplitudes is imposed. Thus, we define phase synchronization of chaotic system as the appearance of a certain relation between the phases of interacting systems (or between the phase of a system and that of an external force), while the amplitudes can remain chaotic and are, in general, non-correlated. Of course, the very notion of phase and amplitude of chaotic systems is rather non-trivial, we will discuss this point in Section 2.

Roughly speaking, the phase of an autonomous self-sustained oscillatory system is related to the symmetry with respect to time shifts. Therefore, the phase disturbances do not grow or decay, what corresponds to the zero Lyapunov exponent. If the oscillations are periodic, the phase rotates nearly uniformly, while in the chaotic case the dynamics of the phase is affected by chaotic changes of the amplitude, so one can expect a Brownian (random-walk-like) behavior of the phase. The diffusion coefficient determines the coherence of the phase. As we shall show in Section 2, one can easily find systems with different levels of the phase coherence. The phase synchronization then appears when a periodic or nearly periodic force is applied with a frequency close to the mean frequency of the phase rotation: the phase of the chaotic system tends to be entrained by the phase of the force, while the internal chaos tries to destroy appearing coherence.

The phenomenon of phase synchronization of chaotic oscillators, as outlined above, has been addressed in [17–22]. In [17,22] interaction of two chaotic systems has been considered, while in [20] this effect has been discussed for an ensemble of coupled chaotic oscillators (see also [23,24], where phase-related effects in networks of chaotic oscillators have been considered). Synchronization by the external periodic force has been investigated numerically [19], as well as experimentally and theoretically [18]. In [18] it has been suggested to detect phase locking via observations of the power spectrum, while in [19] distributions of instantaneous phases have been followed. We shall show further how these approaches can be incorporated into a single framework.

In the present paper we systematically study the entrainment of the phase of a chaotic system by external driving. We discuss several methods to characterize the phase dynamics, including spectral and correlation properties, distribution of the phases, and the ensemble averaging. The last approach, already used in studies of synchronization of nonlinear systems by external noise [25,26], gives a tool for detecting the phase coherence even if the phase itself is not well defined.

The paper is organized as follows. In Section 2 we discuss the notion of the phase for a chaotic system and show how the phase coherence properties can be described. As basic examples we consider the Lorenz and the Rössler systems. In Section 3 we describe tools characterizing the synchronization under external driving. In Section 4 we study numerically synchronization by periodic driving in different chaotic systems. In Section 5 the synchronization by random driving is described. A case when the phase is not well defined is considered in Section 6. We discuss the results in Section 7.

2. Phase of a chaotic system

2.1. Phase and synchronization of periodic oscillations

In this section we recall basic facts on the synchronization of periodic oscillations (see e.g. [27]). Stable periodic oscillations are represented by a stable limit cycle in the phase space, and the dynamics of phase point on this cycle can be represented as

$$\frac{d\phi}{dt} = \omega_0, \tag{1}$$

where $\omega_0 = 2\pi/T_0$, and T_0 is the period of the oscillation. It is important that starting from any monotonically growing variable θ on the limit cycle, one can introduce the phase satisfying Eq. (1). Indeed, an arbitrary θ obeys $\dot{\theta} = \nu(\theta)$ with a periodic $\nu(\theta + 2\pi) = \nu(\theta)$. A change of variables $\phi = \omega_0 \int_0^\theta [\nu(\theta)]^{-1}\, d\theta$ gives the correct phase, where the frequency ω_0 is defined from the condition $2\pi = \omega_0 \int_0^{2\pi} [\nu(\vartheta)]^{-1}\, d\theta$. A similar approach leads to correct angle–action variables in Hamiltonian mechanics. From (1) it is evident that the phase corresponds to the zero Lyapunov exponent, while negative exponents correspond to the amplitude variables.

When a small external periodic force with frequency Ω is acting on periodic oscillations, in the first approximation one can neglect variations of the amplitudes to obtain for the phase.

$$\frac{d\phi}{dt} = \omega_0 + G(\phi, \psi), \quad \frac{d\psi}{dt} = \Omega, \tag{2}$$

where $G(\cdot, \cdot)$ is 2π-periodic in both arguments. The system (2) leads to a circle map with a well-known structure of phase-locking intervals (Arnold's tongues) [27]; each of the intervals corresponds to a synchronization region. This picture is universal and its qualitative features do not depend on the characteristics of the oscillations and of the external force (e.g. nearly sinusoidal or relaxational), and on the relation between amplitudes.

We recall that the synchronization of periodic oscillators can be defined as phase entrainment (locking)

$$|n\phi(t) - m\psi(t)| < \text{const} \tag{3}$$

or a weaker condition of frequency locking

$$\omega = \left\langle \frac{d\phi}{dt} \right\rangle = \frac{m}{n}\Omega, \tag{4}$$

where n and m are integers. In general, both these properties are destroyed in the presence of noise $\xi(t)$, when instead of (2) one has

$$\frac{d\phi}{dt} = \omega_0 + G(\phi, \psi) + \xi(t), \quad \frac{d\psi}{dt} = \Omega. \tag{5}$$

However, if the noise is small, the frequencies are nearly locked, i.e. the averaged relation (4) is fulfilled. Large noise can cause phase slips, i.e. the phase performs random-walk-like motion. In the latter case, although a synchronization region shrinks, strictly speaking, to a point, the largest phase-locking intervals survive as regions of nearly constant mean frequency ω.

A detailed description of synchronization in noise-driven systems is possible if we assume the noise ξ to be Gaussian δ-correlated: $\langle \xi(t)\xi(t')\rangle = 2D\delta(t - t')$. Taking the coupling in the simplest form $G(\phi, \psi) = \varepsilon \sin(\psi - \phi)$ we can reduce the problem to a simple Langevin equation for the phase difference $\theta = \phi - \psi$:

$$\frac{d\theta}{dt} = \Delta - \varepsilon \sin\theta + \xi(t), \tag{6}$$

where $\Delta = \omega_0 - \Omega$. This equation appears also in the theory of Josephson junctions and phase-locked loops [28], its analysis has been performed in [28,29] using the Fokker–Planck equation and in [30] with the help of the cumulant method. We describe here briefly the former approach.

The Fokker–Planck equation reads

$$\frac{\partial W}{\partial \theta} = \frac{\partial}{\partial \theta}[(\Delta - \varepsilon \sin\theta)W] + D\frac{\partial^2 W}{\partial \theta^2}.$$

Looking for a Fourier-representation of the stationary solution

$$W(\theta) = \sum W_k e^{ik\theta},$$

it is easy to find a continuous-fraction representation of the first Fourier-mode (see [28]):

$$W_1 = \frac{1}{2\pi} \Big/ \left(\frac{2i}{\varepsilon}(\Delta - iD) + 1 \Big/ \left(\frac{2i}{\varepsilon}(\Delta - i2D) + 1 \Big/ \left(\frac{2i}{\varepsilon}(\Delta - i3D) + \cdots\right.\right.\right.$$

The main quantities that characterize the synchronization are the averaged frequency $\langle\dot{\theta}\rangle = \Delta + 2\pi\varepsilon \mathrm{Im}(W_1)$ and the Lyapunov exponent $\langle \mathrm{d}\ln\delta\theta/\mathrm{d}t\rangle = -2\pi\varepsilon\,\mathrm{Re}(W_1)$. From the continuous-fraction solution it is clear that only the combinations of the parameters Δ/ε and D/ε are important. Dependencies of the mean frequency and the Lyapunov exponent on these parameters are reported in Fig. 1. The synchronization is nearly perfect for small noise, and is highly smeared for large noise amplitude D. Without noise, the Lyapunov exponent vanishes outside the phase-locking region, and is negative inside it. With noise, the Lyapunov exponent is always negative, but has a minimum at $\Delta = 0$.

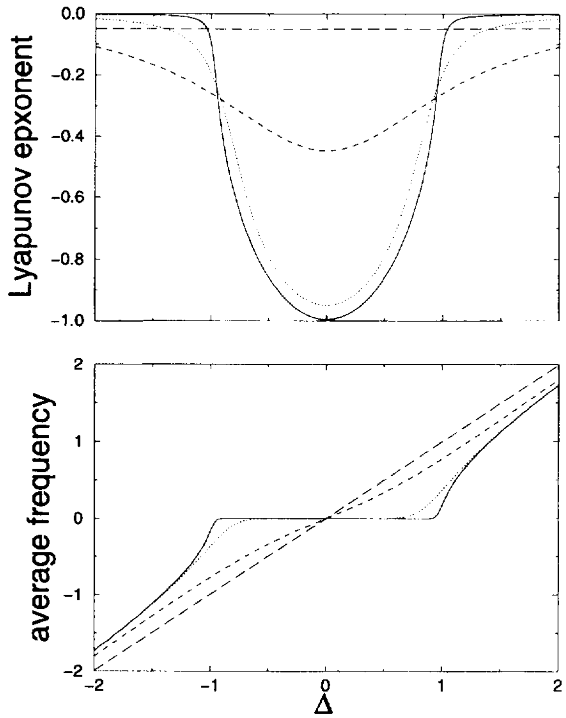

Fig. 1. The Lyapunov exponent and the averaged frequency of the noisy periodic oscillator under external force for $\varepsilon = 1$ and different noise amplitudes (solid line: $D = 0.01$, dotted line: $D = 0.1$, dashed line: $D = 1$, and long-dashed line: $D = 10$).

2.2. Phase of chaotic oscillations

We cannot give an unambiguous and general definition of phase for any chaotic system. Roughly speaking, we want to define phase as a variable that corresponds to the zero Lyapunov exponent of a continuous-time dynamical system which chaotic behavior. In the following we consider three approaches to phase definition:

(A) Sometimes we can find such a projection of the attractor on some plane (x, y) that the plot reminds of the smeared limit cycle, i.e. the trajectory rotates around the origin (or any other point that can be taken as the origin). It means that we can choose the Poincaré section in a proper way. With the help of the Poincaré map we can thus define a phase, attributing to each rotation the 2π phase increase:

$$\phi_{\mathrm{M}} = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \quad t_n \le t < t_{n+1}, \tag{7}$$

where t_n is the time of the nth crossing of the secant surface. Note that for periodic oscillations this definition gives the correct phase satisfying Eq. (1). Defined in this way, the phase is a piecewise–linear function of time. It is clear that shifts of this phase do not grow or decay in time, so it corresponds to the direction with the zero Lyapunov exponent. However, this phase crucially depends on the choice of the Poincaré map, and therefore may disagree with an intuition. We note also that this approach directly corresponds to the special flow construction which is used in the ergodic theory to describe autonomous continuous-time systems [31].

(B) If the above-mentioned projection is found, we can also introduce the phase as the angle between the projection of the phase point on the plane and a given direction on the plane (see also [20,24]):

$$\phi_{\mathrm{P}} = \arctan(y/x). \tag{8}$$

Note that although the two phases ϕ_{M} and ϕ_{P} do not coincide microscopically, i.e. on a time scale less than average period of oscillation, they have equal average growth rates. In other words, the mean frequency defined as the average of $\mathrm{d}\phi_{\mathrm{P}}/\mathrm{d}t$ over large periods of time coincides with a straightforward definition of the mean frequency via the average number of crossings of a Poincaré surface per unit time.

(C) A different way to define the phase is known in signal processing as the analytic signal concept [32]. This general approach, based on the Hilbert transform and originally introduced by Gabor [33], unambiguously gives the instantaneous phase and amplitude for an arbitrary scalar signal $s(t)$. The analytic signal $\zeta(t)$ is a complex function of time defined as

$$\zeta(t) = s(t) + \mathrm{i}\tilde{s}(t) = A(t)\mathrm{e}^{\mathrm{i}\phi_{\mathrm{H}}(t)}, \tag{9}$$

where the function $\tilde{s}(t)$ is the Hilbert transform of $s(t)$

$$\tilde{s}(t) = \pi^{-1}\,\mathrm{P\,V} \int_{-\infty}^{\infty} \frac{s(\tau)}{t - \tau}\,\mathrm{d}\tau \tag{10}$$

(Where P V means that the integral is taken in the sense of the Cauchy principal value). The instantaneous amplitude $A(t)$ and the instantaneous Phase $\phi_{\mathrm{H}}(t)$ of the signal $s(t)$ are thus uniquely defined from (9).

As one can see from (10), the Hilbert transform $\tilde{s}(t)$ of $s(t)$ can be considered as the convolution of the functions $s(t)$ and $1/\pi t$. Hence, the Fourier transform $\tilde{S}(\omega)$ of $\tilde{s}(t)$ is the product of the Fourier transforms of $s(t)$ and $1/\pi t$. For physically relevant positive frequencies $\tilde{S}(\omega) = -\mathrm{i}S(\omega)$; i.e. ideally $\tilde{s}(t)$ may be obtained from $s(t)$ by the filter whose amplitude response is unity, and whose phase response is the constant $\frac{1}{2}\pi$ lag at all frequencies [32,34]. [3]

[3] Practically, for discrete signals given within the bounded time interval, Hilbert transform can be performed with help of a digital filter, e.g. see [34].

A harmonic oscillation $s(t) = A\cos\omega t$ is often represented in the complex notation as $A\cos\omega t + \mathrm{i}A\sin\omega t$. It means that the real-valued oscillation is complemented by the imaginary part which is delayed in phase by $\frac{1}{2}\pi$, that is related to $s(t)$ by the Hilbert transform. The analytic signal is the direct and natural extension of this technique, as the Hilbert transform performs the $-\frac{1}{2}\pi$ phase shift for every frequency component of an arbitrary signal.

Although the analytic signal approach provides the unique definition of the phase of *a signal*, we cannot avoid ambiguity defining the phase for a *dynamical system*, as the result depends on the choice of the observable. Here we face the same problem as in the choice of appropriate projection used for definition of ϕ_M and ϕ_P. However, one can often find an "oscillatory" observable that provides the Hilbert phase ϕ_H in agreement with our intuition. For example, the z-variable is a natural choice for the Lorenz system.

An important advantage of the analytic signal approach is that the phase can be easily obtained from experimentally measured scalar time series, or in other situations when the construction of the Poincaré map is difficult. The comparison of ϕ_M, ϕ_P and ϕ_H for some special examples is given below.

2.3. *Dynamics of the phase of chaotic oscillations*

In contrast to the dynamics of the phase of periodic oscillations, the growth of the phase in the chaotic case cannot generally be expected to be uniform. Instead, the instantaneous frequency depends in general on the amplitude. Let us stick to the phase definition based on the Poincaré map, so one can represent the dynamics as (cf. [18])

$$A_{n+1} = T(A_n), \tag{11}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega(A_n) \equiv \omega_0 + F(A_n). \tag{12}$$

As the amplitude A we take the coordinates on the section surface (if the dimension of the section surface is larger than one, the amplitude should be coinsidered as vector); it does not change during the growth of the phase from 0 to 2π and can be considered as a discrete variable; the transformation T defines the Poincaré map. The phase evolves according to (12), where the "instant" frequency $\omega = 2\pi/(t_{n+1} - t_n)$ depends in general on the amplitude. Assuming the chaotic behavior of the amplitudes, we can consider the term $\omega(A_n)$ as a sum of the averaged frequency ω_0 and of some effective noise $F(A)$ (which in exceptional cases may vanish). Hence, Eq. (12) is similar to the equation describing the evolution of phase of periodic oscillator in the presence of external noise. Thus, the dynamics of the phase is generally diffusive: for large t one expects

$$\langle(\phi(t) - \phi(0) - \omega_0 t)^2\rangle \sim D_\mathrm{p} t,$$

where the diffusion constant D_p determines the phase coherence of the chaotic oscillations. Roughly speaking, the diffusion constant is inversely proportional to the width of spectral peak of the chaotic attractor [35]. Comparing with (5), we can say that the term $F(A)$ can be interpreted as effective phase noise, and the diffusion constant D_p measures the strength of this noise. It is a particular characteristic of chaotic oscillations which does not coincide with usual ones, e.g. the Lyapunov exponents; it does not exist for general discrete-time dynamical systems. We will see that the value of D_p (more precisely, the dimensionless ratio D_p/ω) is crucial for the synchronization properties of chaotic oscillations.

Generalizing Eq. (12) in the spirit of the theory of periodic oscillations to the case of periodic external force, we get in the most general case (the only assumption is that the Poincaré map construction used for the autonomous system is still valid in the non-autonomous case; we expect this to hold at least for weak forcing)

$$A_{n+1} = T(A_n, \psi_n), \tag{13}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \omega(A_n, \phi, \psi), \tag{14}$$

$$\frac{d\psi}{dt} = \Omega. \tag{15}$$

If we assume that the forcing amplitude is small and proportional to ε, we can rewrite Eq. (14) in the first approximation as

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon G(A_n, \phi, \psi) + F(A_n). \tag{16}$$

This equation is similar to Eq. (5) with the amplitude-depending part of the instant frequency playing the role of noise.[4] Thus, we expect that in general the synchronization phenomena for periodically forced chaotic system are similar to those in noisy driven periodic oscillations. One should be aware, however, that the "noisy" term $F(A)$ can hardly be explicitly calculated, and for sure cannot be considered as a Gaussian δ-correlated noise as is usual in the statistical approaches [29,28].

2.4. Phase of chaotic oscillations: Examples

In this section we introduce two chaotic systems that serve as prototype models for the subsequent study and discuss their phase coherence properties. In the following sections we consider these systems under periodic or noisy forcing $\mathcal{F}$; firstly we describe the autonomous case $\mathcal{F} = 0$.

The equations of the Rössler system have the form [36]

$$\dot{x} = -y - z + \mathcal{F}, \qquad \dot{y} = x + 0.15y, \qquad \dot{z} = 0.4 + z(x - 8.5). \tag{17}$$

This attractor has a sharp peak in the power spectrum and a rather simple form (Fig. 2). Here the Poincaré map can be easily constructed, and all the three definitions of the phase give similar results (Fig. 2). In fact, we have found that the difference between ϕ_M, ϕ_P, and ϕ_H is negligible. The diffusion constant for the attractor shown in Fig. 2 is extremely small ($D_p < 10^{-4}$), what corresponds to an extremely sharp peak in the spectrum (see also discussion in [37]). Thus, this attractor can be called phase coherent.

The second model we consider in this paper is the Lorenz system [38]

$$\dot{x} = 10(y - x), \qquad \dot{y} = 28x - y - xz, \qquad \dot{z} = -\tfrac{8}{3}z + xy + \mathcal{F}. \tag{18}$$

Due to the symmetry $x \to -x, y \to -y$ we take the projection on the plane spanned by the variables z and $u = \sqrt{x^2 + y^2}$ (Fig. 3); and define the instantaneous phase as

$$\phi_P(t) = \arctan\left(\frac{u(t) - u_0}{z(t) - z_0}\right), \tag{19}$$

where $z_0 = 27$, $u_0 = 12$ are the coordinates of the equilibrium point. Here the three definitions of the phase give similar results, like for the phase coherent Rössler attractor above. However, the diffusion constant is not very small: $D_p \approx 0.2$ when the phase ϕ_P is used, to be compared to the mean frequency $\omega \approx 8.3$. Thus, although the motion is topologically simple, effective noise in the phase dynamics is relatively large (due to the obvious non-uniformity of the motion in the vicinity of the saddle fixed point at the origin).

Concluding this section, we can say that the systems described above provide examples of attractors with high and moderate phase coherence, respectively. In other words they have, respectively, small and moderate effective internal noise. We will see that their response to the periodic external drive is also different.

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[4] In the case of periodic oscillations the amplitude is nearly constant, therefore the function G in (5) depends only on phase variables.

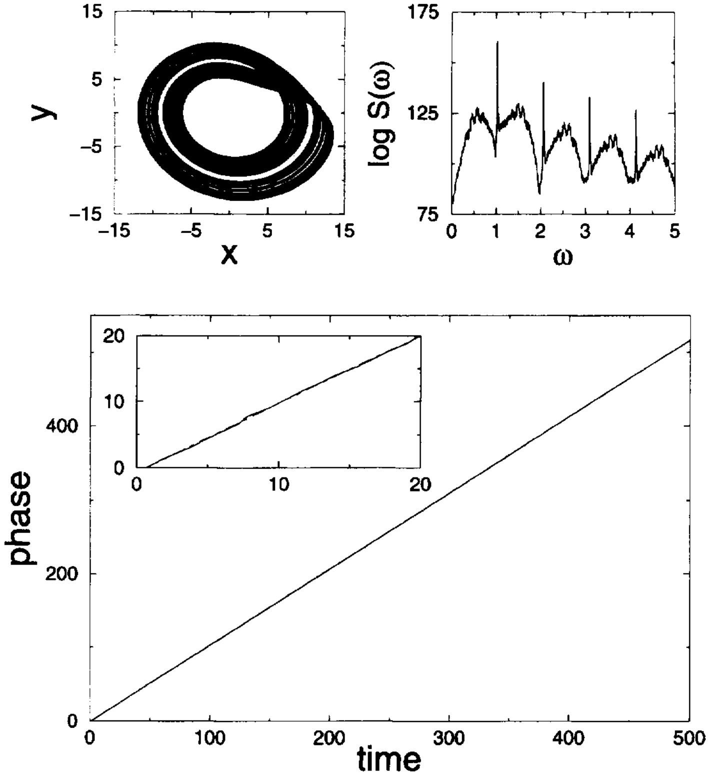

Fig. 2. The phase portrait in the coordinates (x, y), the power spectrum of $x(t)$, and the time evolution of the phases for the phase-coherent Rössler oscillator (solid line: the phase ϕ_P, dashed line: the Hilbert-transform phase ϕ_H, and dot-dashed line: the phase ϕ_M).

3. Characterizing phase synchronization by external force

3.1. Phase and frequency locking

If the phase of a chaotic oscillator is well-defined, i.e. all approaches to the definition of the phase give similar results, we can use the same criteria of synchronization as for periodic oscillations ((3) and (4)). If relation (3) is fulfilled, one can speak on the phase locking of chaotic oscillations. In the case of relatively large effective noise in the phase dynamics, the phase locking seems to be rather exceptional, so the weaker condition of frequency locking (4) should be used. We emphasize that the mean frequency of chaotic oscillations ω can be calculated rather easily: as it follows from (7)

$$\omega = \lim_{t\to\infty} 2\pi \frac{N_t}{t}, \tag{20}$$

where N_t is the number of crossings of the Poincaré section during observation time t. This method can be straightforwardly applied to the observed time series, in the simplest case one can e.g. take for N_t the number of maxima of $x(t)$. Note that the frequency 20 does not necessarily coincide with the maximal frequency in the power spectrum, although we expect them to be close in topologically simple attractors.

It would be, however, useful to have such characteristics of synchronization that do not depend on the definition of the phase and where one does not need to compute it explicitly. As we will see in Section 6, in some cases the phase

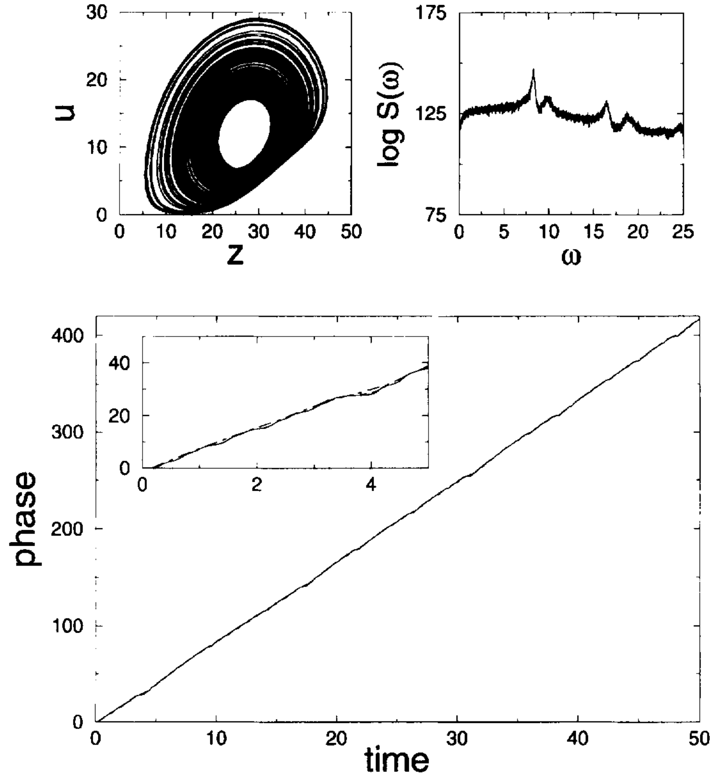

Fig. 3. The phase portrait in the coordinates (z, u), the power spectrum of $z(t)$, and the time evolution of the phase for the Lorenz oscillator (solid line: the phase ϕ_P, dashed line: the Hilbert-transform phase ϕ_H, and dot-dashed line: the phase ϕ_M).

is not a well-defined quantity, and therefore we need some tools that can characterize the phase synchronization indirectly. We discuss here several such approaches.

3.2. *Probability distribution*

The invariant measure of an autonomous chaotic system gives a nearly uniform distribution of the phases. With a periodic external force, the measure is explicitly time-dependent. Phase synchronization means that for each time the distribution density of the phases is non-uniform (there is a time-dependent preferable value of the phase), where the sharpness of the peak characterizes the level of synchronization. This peak in the phase distribution rotates with the phase of the external force. The distribution of the amplitudes, however, remains broad. Due to ergodicity, the probability distribution can be obtained also from a chaotic trajectory, if it is observed stroboscopically in the proper phase of external force.

3.3. *Dynamics of ensemble average*

The distribution of the phase as discussed above can be considered as a distribution in an ensemble of N identical uncoupled chaotic systems driven by the same periodic force. Let us characterize this ensemble with the ensemble average of an observable u, which we assume to be phase-dependent (otherwise the phase synchronization effects will be not seen):

$$U(t) = \frac{1}{N}\sum_{1}^{N} u_i. \tag{21}$$

In the case when the phase is not entrained by the phase of the external force, we can consider it as arbitrary and nearly uniformly distributed from 0 to 2π. Thus, the ensemble average consists of non-coherent contributions and is nearly constant (apart of finite-size fluctuations of order $N^{-1/2}$). Contrary to this, if the phases are locked by external force, the contributions to the ensemble average are coherent and one observes macroscopic oscillations of the field $U(t)$ with the frequency of the external force. Thus, the transition to phase synchronization manifests itself in the appearance of a macroscopic average field in an ensemble of identical periodically driven chaotic systems. [5]

The ensemble average can be directly related to the time-dependent probability distribution W. In the periodically forced system the invariant probability distribution is a periodic function of time (we consider only the distribution of the observable u which we use for the average (and for the spectrum, see below) calculations):

$$W(u, t - t_0) = W(u, t + T - t_0). \tag{22}$$

Here we have also introduced the explicit dependence on the phase of the external force which is given by the initial time t_0. The ensemble average in the thermodynamic limit ($N \to \infty$) can be written as

$$U(t - t_0) = \int \mathrm{d}u \, u W(u, t - t_0). \tag{23}$$

It is a periodic function of time and depends on the initial moment of time t_0. The variance of the oscillations $V = \overline{(U - \overline{U})^2}$ serves thus as a measure of phase synchronization. It is important that for this measure we do not have to calculate the phase itself, i.e. it is independent of the phase definition.

2.4. *Discrete component in the spectrum*

Another indirect characteristic of the phase locking has been proposed in [18]. If the phase of a chaotic system is locked by periodic force, the process becomes highly correlated in time: the values of an observable u at times t and $t + nT$ (we recall that T is the period of the external force) differ only due to the chaotic nature of the amplitudes, because the phases at these times are almost identical. This can be seen by calculating the autocorrelation function $C(\tau) = \langle u(t)u(t+\tau)\rangle$ which has a periodic tail for $\tau \to \infty$ with maxima at $\tau = nT$. Thus, in the power spectrum high δ-peaks appear at the frequency of external force Ω and its harmonics $n\Omega$. Therefore, one can characterize the phase synchronization by calculating the discrete part of the power spectrum. An appropriate quantity is the intensity of the discrete spectrum defined according to the Wiener lemma [31,39] as

$$S = \lim_{t\to\infty} \frac{1}{t} \int_0^t C^2(\tau)\,\mathrm{d}\tau. \tag{24}$$

A resonance-like curve S vs. Ω is an indicator of the phase synchronization (and, like ensemble average, it is independent on the definition of the phase).

It is noteworthy that the discrete component in the spectrum can be calculated from the time-dependent probability distribution density and related to the intensity of variations of the ensemble average, see Appendix A.

[5] A macroscopic field can also appear in an ensemble of non-identical systems, see [20].

One important remark should be made concerning the discrete spectrum and the ensemble average. Because the distribution density is a periodic function of time (22), the appearance of non-zero ensemble averages and of the discrete components in the spectrum is a general property of periodically forced systems, regardless of the synchronization phenomenon. The main idea of using these characteristics when describing synchronization is that in the synchronous case the distribution density becomes extremely sharp in phase. As a result, the ensemble average and the intensity of discrete spectrum demonstrate a clear and in some cases large maximum in dependence on the frequency of the external force (see Figs. 6 and 7).

3.5. *Lyapunov exponents*

The phase entrainment of *periodic* oscillators can be characterized by means of the largest Lyapunov exponent: [6] phase locking corresponds to a negative exponent, while quasiperiodic motion between Arnold tongues has a zero exponent. For chaotic systems, the largest Lyapunov exponent is positive, and in the autonomous case phase shifts correspond to the zero Lyapunov exponent. When an external force is applied, the degeneracy disappears and the zero exponent can become positive or negative. In the phase-locked state we expect this exponent to be negative, corresponding to a stable (time-dependent) value of the phase in relation to the phase of the external force. Outside of the synchronization region the absolute value of the exponent should be small; however, we do not expect that it will be exactly zero. Numerical results in Section 4 show that it can be both positive and negative. Thus, we cannot use the Lyapunov exponents as definite criterion for phase synchronization. This is in contrast to other types of synchronization, where the transition point is determined as a point where the corresponding Lyapunov exponent changes its sign [7,8,40].

4. Phase synchronization by harmonic external force

Throughout this section the external force in Eqs. (17) and (18) is considered to be harmonic, i.e. $\mathcal{F} = E\cos(\Omega t)$.

4.1. *Rössler attractor*

We study numerically the synchronization of the phase-coherent Rössler system (17). The main frequency ω is calculated using phase definitions from the Hilbert transform (9) and from (8) with nearly coinciding results. The dependence of ω on the amplitude E and the frequency of the external force Ω (Fig. 4) demonstrates clearly that there exist phase-locking regions which correspond to the main resonance $\Omega \approx \omega_0$ (Fig. 4(a)) and to the resonances $\Omega \approx 2\omega_0$ (Fig. 4(b)) and $2\Omega \approx \omega_0$ (Fig. 4(c)). It is worth noting that there seems to be no threshold of synchronization; we attribute this to the extremely high phase coherence of the Rössler attractor. In other words, the effective internal noise is very small and the phase behavior is not far from that of a periodic self-oscillating system.

Next, we study how the probability distribution of phases changes with the onset of synchronization. As we have discussed in Section 3, there are two ways of getting this distribution: either to consider evolution of a large ensemble, or to make a stroboscopic plot of the states of a single system. The second method is obviously simpler and we use it to obtain the instantaneous distribution density $W(x, y, t)$ (Fig. 5). Inside the main resonance region (Fig. 5(a)) the phases are locked, while the distribution of the amplitudes remains broad. Chaoticity of the amplitude dynamics can be easily checked by plotting subsequent amplitudes as shown in Fig. 5 (right column): the one-hump

[6] We do not consider the Lyapunov exponent corresponding to the external force.

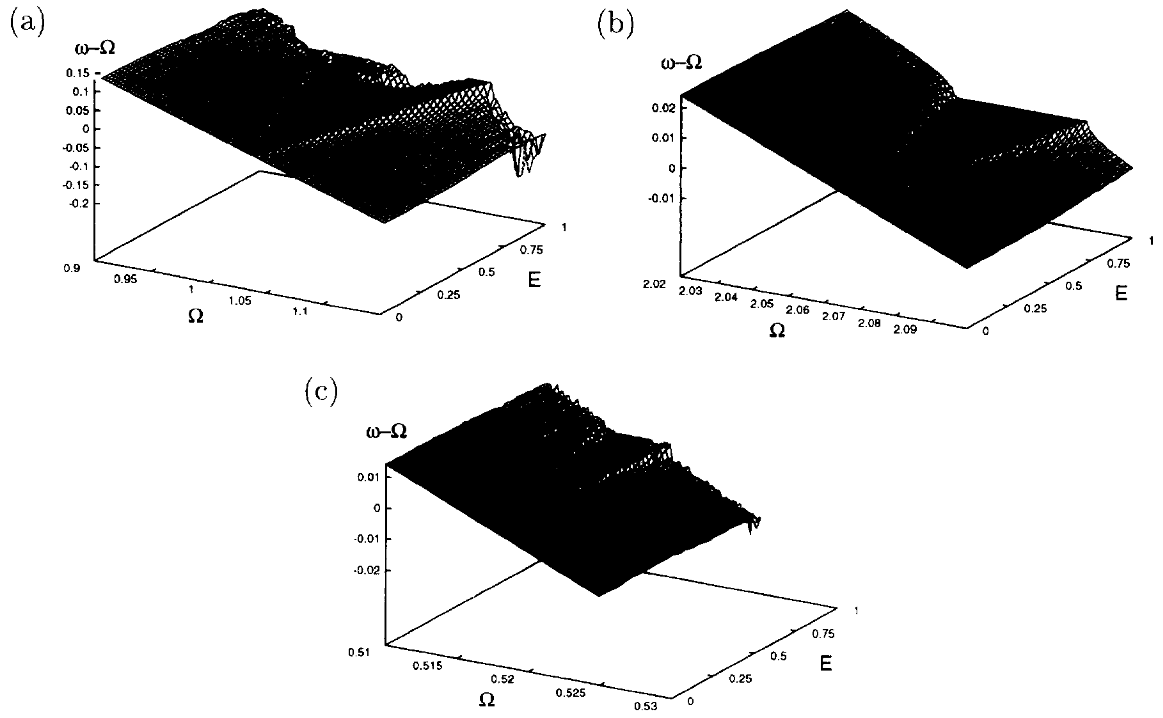

Fig. 4. The "Arnold tongues" in the forced Rössler system: $\omega - \Omega$ vs. the amplitude E and the frequency Ω of the external force.

mapping corresponds to the stretching and folding of the trajectories in the Rössler system. Outside of the resonance region (Fig. 5(b)) the phases are distributed from 0 to 2π, although seemingly non-uniform. At the resonance 1 : 2 (Fig. 5(c)) there are two stable values of the phase for each instant of time. The cluster Fig. 5(a) rotates with the frequency Ω, so the macroscopic average field (21) appears oscillating with frequency Ω. Outside the resonance (Fig. 5(b)) this field is nearly zero.

The dependence of the average field variance in the ensemble of identical systems on the driving frequency (Fig. 6(a)) shows a large peak at $\Omega \approx \omega$. The very narrow-band peak at $\Omega \approx 1.06$ corresponds to the periodic windows inside chaotic region. This is confirmed with the help of the Lyapunov exponents calculation (Fig. 6(b)). Note that the phase and the frequency can be equally defined both for chaotic and regular oscillations, and because in the phase-coherent case the dynamics of the phase is almost independent of that of the amplitude, one cannot distinguish non-chaotic windows inside and outside the frequency locking region from the frequency observations presented in Fig. 4.

4.2. *Lorenz attractor*

We report here the results for the periodically forced Lorenz system (Eq. (18)). We were not able to find the resonances 1 : 2 and 2 : 1, and the synchronization at the main resonance $\Omega = \omega$ appears only for sufficiently large amplitudes of the external force. This can be explained by the presence of a relatively large noise in the dynamics of the phase (see Section 2).

We present the results for different amplitudes of the external drive in Fig. 7. The plateau appearing in the dependence of $\Omega - \omega$ vs. Ω is not perfect, as one can expect for the synchronization in the presence of noise

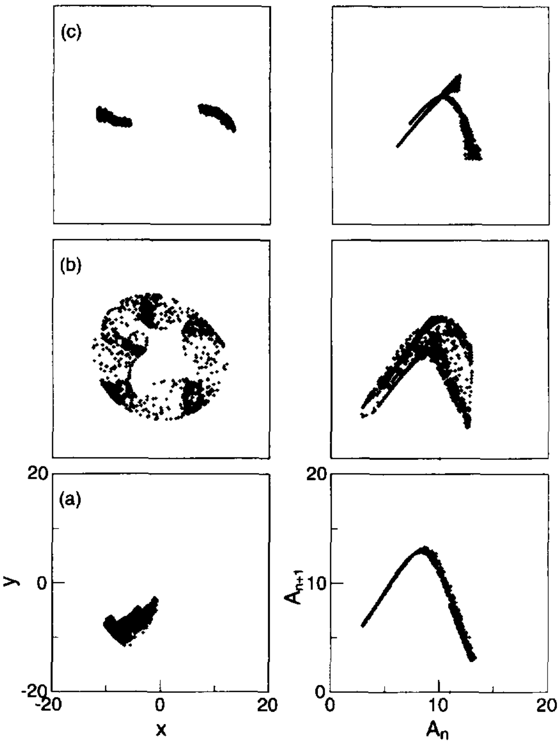

Fig. 5. Left column: the distribution of states of Rössler oscillators projected on the (x, y) plane; for $E = 0.5$ and different driving frequencies: (a) $\Omega = 1.03$; the phases are perfectly locked. (b) $\Omega = 1.3$; no synchronizationn, the phases are distributed from 0 to 2π. (c) $\Omega = 2.06$; 1 : 2 resonance, there are two stable states for phases. In this case the ensemble average of x or y vanishes, so another observable (e.g. x^2) should be chosen for ensemble average calculations. Right column: the amplitudes $A_n = (x^2 + y^2)^{1/2}$ taken at the times $t = n \cdot 2\pi/\Omega$.

(cf. Fig. 1). Contrary to the phase-coherent case, the region of the negative second Lyapunov exponent [7] seems not to coincide exactly with the region of synchronization. However, for small and moderate amplitudes of the external force the behavior of the second Lyapunov exponent is similar to that in the noisy driven periodic oscillator (Fig. 1). The correspondence is bad for large amplitudes of external force, where presumably the structure of the attractor changes significantly.

The behavior of the average field in an ensemble of Lorenz systems, and the intensity of the discrete component in the spectrum calculated to (24) are also reported in Fig. 7. Both curves demonstrate a clear maximum when the external frequency Ω is close to the mean frequency of oscillations ω. For large amplitudes of the external force the form of the maximum is rather complex (see also Fig. 4(a)); here the phase dynamics apparently cannot be separated from that of the amplitude.

5. Phase synchronization by external noisy drive

We have discussed above the phase synchronization by periodic external drive. In this section we show that similar effects occur when the forcing is noisy. Synchronization of periodic oscillators by external noise have been

[7] The largest Lyapunov exponent remains positive in this region (and therefore is not shown) – there are no windows of periodic behavior.

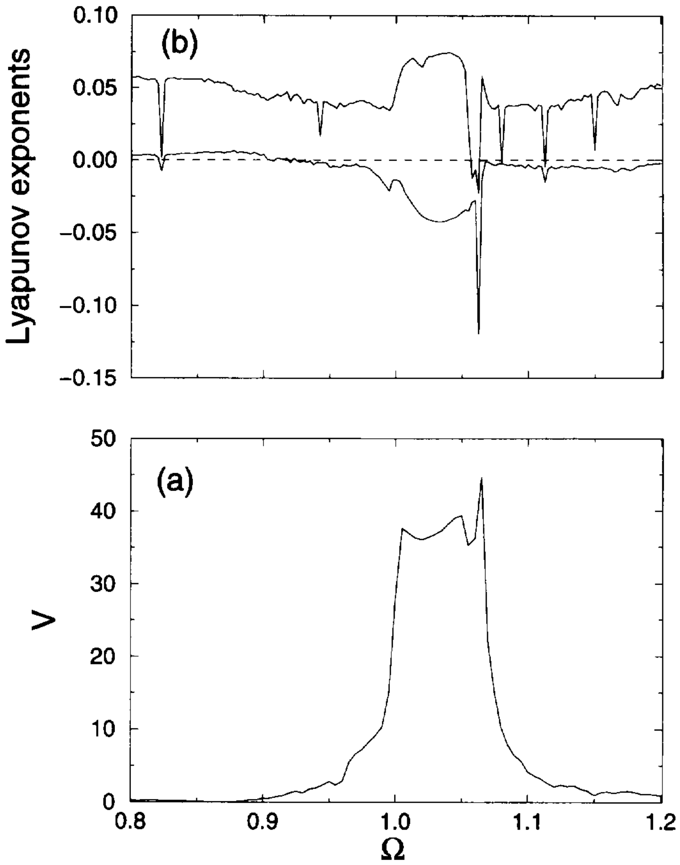

Fig. 6. The variance of the ensemble average field (with x as an observable) for the ensemble of $N = 5000$ Rössler oscillators vs. driving frequency Ω for $E = 0.5$. (b) The first and the second Lyapunov exponents.

considered in [25,26]. It has been shown there that in presence of noise the largest Lyapunov exponent can become negative. The resulting synchronization, however, has not been analyzed from the viewpoint of phase entrainment or frequency locking, only the coincidence of states of an ensemble of oscillators driven by the same noise has been demonstrated [26]. Thus, from all characteristics of synchronization discussed in Section 3, only the calculation of the mean field has been considered.

Generalizing this approach to the case of noisy driven chaotic oscillators, one can expect that under the influence of noise the zero Lyapunov exponent corresponding to the phase dynamics can become negative, so the phases in an ensemble of systems driven by the same noise will tend to coincide (or nearly coincide) resulting in the appearance of a macroscopic mean field in the large ensemble.

To analyze the phase synchronization by noise, we start from the equation for the phase (12) and assume that the external force is a sequence of δ-pulses occurring at random times t_i with an amplitude $Q(\phi)$:

$$\frac{d\phi}{dt} = \omega_0 + F(A) + Q(\phi)\sum \delta(t - t_i).$$

If we denote the phase before the ith pulse as ϕ_i, we get

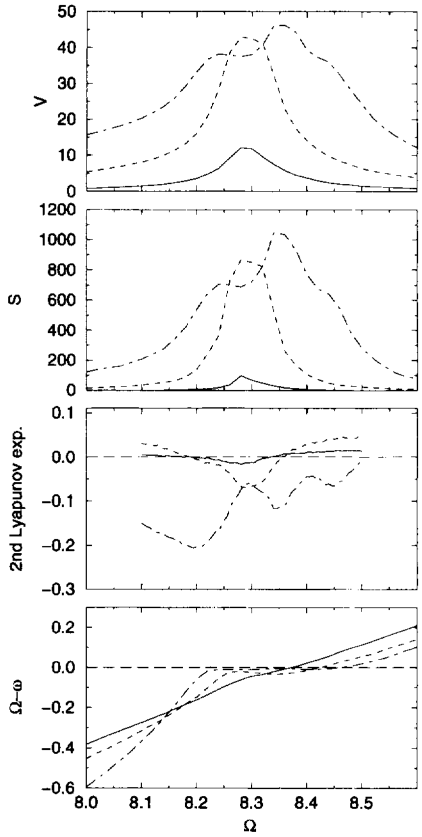

Fig. 7. Phase synchronization in the Lorenz system for different amplitudes of the periodic force: $E = 2$ (solid line), $E = 5$ (dashed line), and $E = 10$ (dot-dashed line). The graphs show the observed frequency ω, the second Lyapunov exponent, the intensity of the discrete spectrum S and the variance of the mean field in an ensemble of 5000 systems V vs. the driving frequency Ω.

$$\phi_{i+1} = \phi_i + \int_{t_i}^{t_{i+1}} (\omega_0 + F(A))\,\mathrm{d}t + g(\phi_i), \tag{25}$$

where

$$g(\phi) = q^{-1}(1 + q(\phi_i)), \quad q'(\phi) = \frac{1}{Q(\phi)}.$$

This mapping (25) is a noisy nonlinear circle map, of the type considered in [25,26,41]. There are two sources of noise in this map: one is the chaotic time-dependence of the amplitudes of the oscillations, and the second one is the randomness of the time interval between pulses $t_{i+1} - t_i$. If the noisy term is not too large, it does not destroy the phase synchronization, as we have already discussed in Section 2. It is important to note that although the phase in Eq. (25) is synchronized, it does not mean appearance of long correlations or discrete spectrum, because the phase is entrained by the phase of noise, which itself is a random function of time.

We illustrate the effect of phase synchronization by external noise with the forced Rössler system

$$\begin{aligned} &\dot{x} = -y - z + E\cos(\psi), \qquad \dot{y} = x + 0.15y, \\ &\dot{z} = 0.4 + z(x - 8.5), \qquad \dot{\psi} = \Omega + \eta, \qquad \dot{\eta} = D(-\eta + \epsilon\xi(t)), \end{aligned} \tag{26}$$

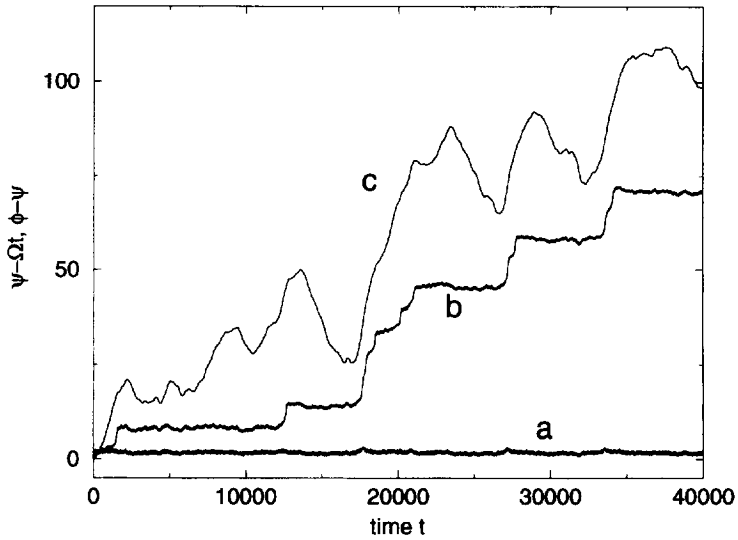

Fig. 8. The phase difference $\phi - \psi$ (curves a,b) and the phase of the external drive $\psi - \Omega t$ (curve c) for $D = 0.001$, $\Omega = 1.03$, $\epsilon = 0.5$ and different amplitudes of the noisy force: (a) $E = 0.5$, (b) $E = 0.25$.

where $\xi(t)$ is Gaussian δ-correlated noise $\langle \xi(t)\xi(t')\rangle = \delta(t - t')$. Here the external force $E\cos(\psi)$ is a process with the random phase ψ which has a continuous power spectrum with a peak at frequency Ω. The width of the peak is proportional to the parameter D in the equation for the auxiliary variable η. In order to prove the synchronization, we plot in Fig. 8 the difference between the phase of the Rössler oscillator ϕ, defined according to the relation (8), and the phase of the external noise ψ. This difference is almost constant, while the phase ψ demonstrates a random-walk motion. One can expect full phase synchronization if the characteristic time scales of this random-walk motion lie inside the phase-locked region Fig. 4, in the case of faster motions phase slips occur (curve b in Fig. 8). We have observed similar effects if the drive is chaotic (cf. [22]).

6. Systems with not well-defined phase

As we have discussed above, definition of the phase implies some restrictions on the topological properties of the attractor. If these restrictions are not fullfilled, we can say that the phase is ill- defined (although in autonomous dynamical systems one Lyapunov exponent is exactly zero, and this can be interpreted as existence of a phase). Consider e.g. the Rössler system for parameters slightly different from those in (17): instead of the term $0.15y$ take $0.25y$ in the second equation. This change leads to the appearance of the so-called funnel attractor, shown in Fig. 9. The topological structure is now complex: there are small and large loops on the x, y plane, and it is not clear which phase shift (π or 2π) should be attributed to these loops. Respectively, different definitions of the phase give different results. Nevertheless, we have applied an external force to this system (in the same way as to the phase-coherent Rössler system) and looked for possible effects of phase synchronization. Because the phase itself is ill-defined, we considered only implicit characteristics of synchronization, such as the ensemble average field and the discrete component in the spectrum. The results are presented in Fig. 10. One can see that there exists a range of external frequencies where both the quantities have a maximum, although this maximum is much smaller than for the phase-coherent Rössler attractor (Fig. 6). In the region of this maximum the second Lyapunov exponent is

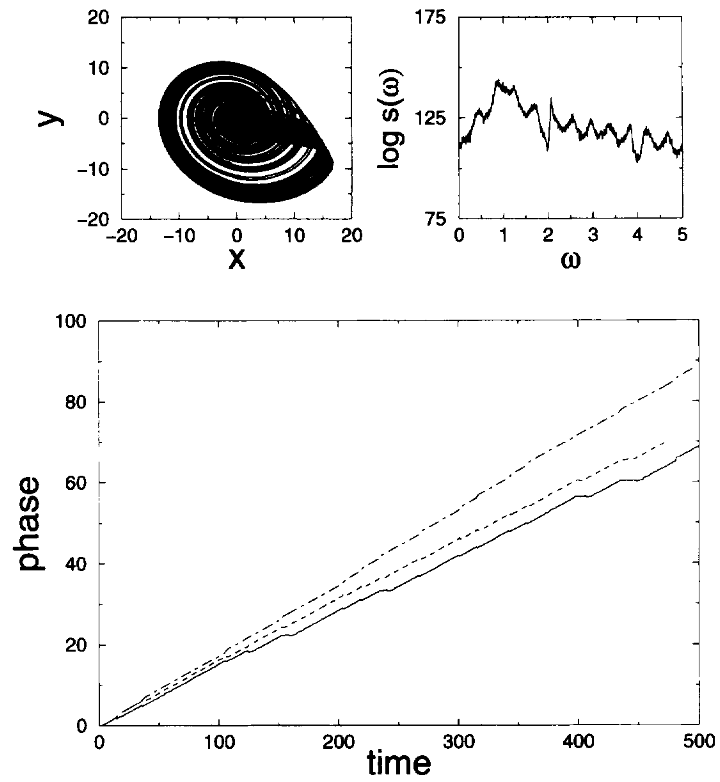

Fig. 9. The same as Fig. 2, but for the funnel Rössler oscillator.

rather small in the absolute value, fluctuating around zero. We suggest to interpret these results as manifestation of the phase synchronization phenomenon in a system with very large effective noise (the noise in "phase" dynamics coming from the switching between large and small loops). This is consistent with findings of Ref. [20], where the Kuramoto-type self-synchronization transition has been observed both for ensembles of phase-coherent and funnel Rössler attractors.

7. Discussion

The main idea of this paper is to extend the notion of phase to the case of chaotic oscillators. Although we are not able to do this rigorously and to suggest a unique definition of the phase, we have shown that it can be introduced in some reasonable and consistent way for different chaotic oscillators. We have proposed and compared three approaches to the definition of the phase. The dynamics of the phase of a chaotic system is similar to that of periodic oscillator in the presence of noise. Here, the chaotic behavior of amplitudes acts as some effective noise, although it is of purely deterministic origin. Therefore, we expect that many, if not all, synchronization features known for periodic oscillators can be observed for chaotic systems as well.

In this paper we have studied the effect of phase synchronization of a chaotic self-sustained oscillator by external drive. In the synchronous regime, the phase or frequency of the oscillator is entrained, respectively, by the phase or frequency of the force, while the amplitudes of the oscillator remain chaotic. As we have demonstrated, this effect can

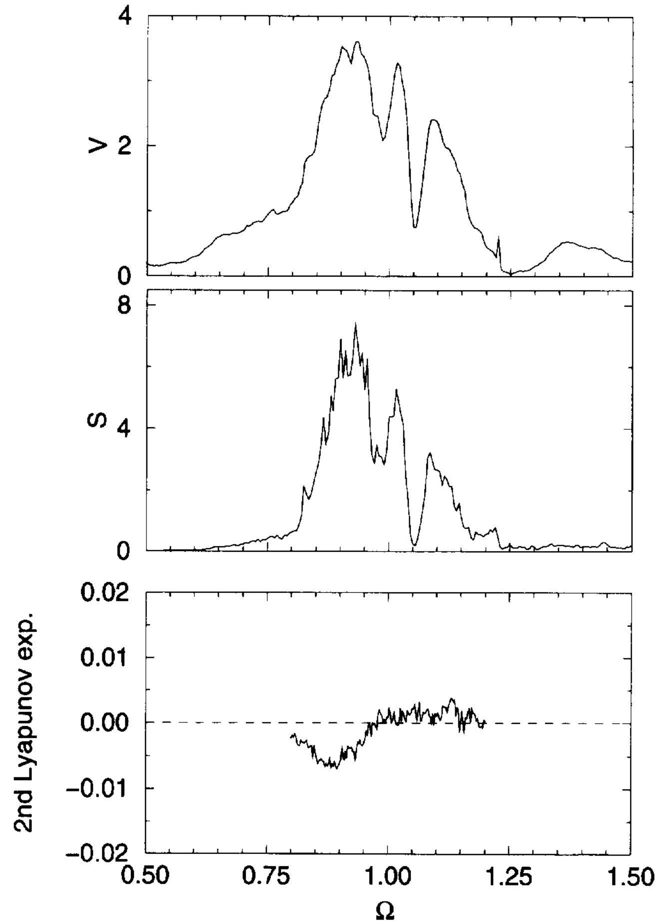

Fig. 10. The variance of the ensemble average field (with x as an observable) for the ensemble of $N = 5000$ funnel Rössler oscillators vs. driving frequency Ω for $E = 0.5$.

appear for periodic, chaotic and noisy forcing. It appears to be robust enough to be observed experimentally [18,22]. Among other phase synchronization phenomena already studied for chaotic oscillators we mention the effect of mutual synchronization of two non-identical chaotic oscillators [17], self-synchronization in a large population of globally coupled oscillators [20], and phase synchronization in a lattice of chaotic oscillators [21].

It is noteworthy that even in the case when the phases are not well defined, the presence of phase synchronization can be demonstrated indirectly, i.e. independently of any particular definition of the phase. In particular, in large ensembles it manifests itself macroscopically in the appearance of the non-zero ensemble average due to the phase entrainment. This effect may be important e.g. for the understanding of different phenomena of collective behavior in ensembles of neurons [42].

We believe that the described effect of phase synchronization can find a number of practical applications, especially if a coherent summation of signals from several chaotic oscillators is desirable to get a high-power output (e.g. in laser arrays [43,44]). As the phase synchronization can be achieved by a rather weak external action, it can be used for controlling the output in a network of chaotic oscillators.

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Appendix A

Here we derive relation between the intensity of the discrete spectrum and the variance of the ensemble average. To calculate the autocorrelation function of the observable u using the time-dependent distribution (22) we have to average over the uniform distribution of t_0 as well:

$$C(\tau) = \overline{\langle u(t)u(t+\tau)\rangle} = \frac{1}{T^2}\int_0^T \mathrm{d}t_0 \int_0^T \mathrm{d}t \int \mathrm{d}u\,\mathrm{d}v\, uv W_2(u, t-t_0; v, t-t_0+\tau),$$

where $W_2(u, t-t_0; v, t-t_0+\tau)$ is the two-point probability density. For a chaotic system, for large τ due to mixing property (which is not complete, because the probability density depends explicitly on time) we have

$$W_2(u, t-t_0; v, t-t_0+\tau) \approx W(u, t-t_0) \cdot W(v, t-t_0+\tau).$$

Therefore, for large τ

$$C(\tau) \approx \frac{1}{T^2}\int_0^T \mathrm{d}t_0 \int_0^T \mathrm{d}t\, U(t-t_0)U(t-t_0+\tau) = C_U(\tau),$$

where $C_U(\tau)$ is the correlation function of the periodic process $U(t)$ (see Eq. (23)). The intensity of the discrete spectrum in the process $u(t)$ is, according to the Wiener's lemma [31,39],

$$S = \lim_{t\to\infty} \frac{1}{t}\int_0^t C^2(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} C^2(\tau)\,\mathrm{d}\tau = \frac{1}{T}\int_t^{t+T} C_U^2(\tau)\,\mathrm{d}\tau.$$

Representing the mean field as Fourier-series

$$U = \sum_{k=-\infty}^{+\infty} A_k \mathrm{e}^{\mathrm{i}k\Omega t},$$

its correlation function can be written as

$$C_U(\tau) = \sum_{k\neq 0} |A_k|^2 \mathrm{e}^{\mathrm{i}k\Omega\tau}.$$

Thus, we get for the intensity of the discrete spectrum

$$S = \sum_{k\neq 0} |A_k|^4$$

and for the intensity of the mean field

$$V = \overline{(U-\overline{U})^2} = C_U(0) = \sum_{k\neq 0} |A_k|^2.$$

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