Phase Synchronization in Driven and Coupled Chaotic Oscillators

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Abstract— **We describe the effect of phase synchronization of chaotic oscillators. It is shown that phase can be defined for continuous time dynamical oscillators with chaotic dynamics, and effects of phase and frequency locking can be observed. We introduce several tools which characterize this weak synchronization and demonstrate phase and frequency locking by external periodic force, as well as due to weak interaction of non-identical chaotic oscillators. In the synchronous state the phases of two systems are locked, while the amplitudes remain chaotic and non-correlated. The intermittency phenomenon at the synchronization transition is considered. The application to the analysis of bivariate experimental data is discussed.**

Keywords— **chaotic oscillators, phase dynamics, frequency locking, weak interaction.**

I. Introduction

SYNCHRONIZATION is a basic phenomenon in physics, discovered at the beginning of the modern age of science by Huygens [1]. In the classical sense, synchronization means adjustment of frequencies of periodic oscillators due to a weak interaction (cf. [2], [3]). This effect is well studied and finds a lot of practical applications in electrical and mechanical engineering [4].

Recently, with widespread studies of chaotic oscillations, the notion of synchronization has been generalized to the latter case. In this context, various phenomena have been found which are usually referred to as "synchronization", so one needs a more precise description to specify them. So, periodic external force acting on a chaotic system can destroy chaos, and a periodic regime appears [5]. This effect can be referred to as "chaos–destroying" synchronization. Due to a strong interaction of two (or a large number) of identical chaotic systems, their states can coincide, while the dynamics in time remains chaotic [6], [7]. This case can be denoted as "complete synchronization" of chaotic oscillators. It can be easily generalized to the case of slightly non-identical systems [7] or the interacting subsystems [8]. A different approach is based on the calculation of the attractor dimension of the whole system and its comparison with the partial dimensions calculated in the phase subspaces formed by the coordinates of each interacting oscillator [9], [10]. In refs. [11], [12], [13] synchronization in chaotic systems has been defined as overlapping of power spectra of respective signals. A generalized synchronization [14], [15], introduced for drive–response systems, is defined as the presence of some functional relation between the states of response and drive, i.e. $\mathbf{x}_2(t) = \mathcal{F}(\mathbf{x}_1(t))$ [16]. All these phenomena occur for a relatively strong forcing, and their characteristic feature is the existence of a threshold coupling value (depending on the Lyapunov exponents of the individual systems) [6], [7], [17], [18], [19].

In this paper we systematically describe the effect of *phase synchronization* of chaotic systems due to weak interaction or external forcing. This phenomenon is mostly close to synchronization of periodic oscillations, where only the phase locking is important, while no restriction on the amplitude is imposed [20], [3]. Thus, we define phase synchronization of chaotic system as the occurrence of a certain relation between the phases of interacting systems (or the phase of a system and that of an external force), while the amplitudes can remain chaotic and are, in general, uncorrelated. Of course, the very notion of phase and amplitude of chaotic systems is rather non-trivial.

Roughly speaking, the phase of an autonomous self-sustained oscillatory system is related to the symmetry with respect to time shifts. Therefore, the phase disturbances do not grow or decay, what corresponds to the zero Lyapunov exponent. If the oscillations are periodic, the phase rotates nearly uniformly, while in the chaotic case the dynamics of the phase is effected by chaotic changes of the amplitude, so one can expect a Brownian (random–walk–like) behavior of the phase. The diffusion coefficient determines the coherence of the phase. As we shall show below in section II, one can easily find systems with different levels of phase coherence. The phase synchronization appears when a periodic or nearly periodic force is applied with a frequency close to the mean frequency of the phase rotation. The phase of a chaotic system tends to be entrained by the phase of the force (or that of another oscillator), while the internal chaos tries to destroy the appearing coherence.

II. Phase of chaotic oscillations

We cannot give an unambiguous and general definition of phase for chaotic systems. Nevertheless, we propose different approaches that allow us to describe in a reasonable way the phase and frequency locking phenomena in chaotic systems.

A. Determination of the phase of chaotic systems

A.1 Based on the Poincaré map

Sometimes we can find a projection of the attractor on some plane (x, y) such that the plot looks like a smeared limit cycle, i.e. the trajectory rotates around the origin. This means that we can choose a Poincaré section in a

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proper way. With the help of the Poincaré map we can thus define a phase, attributing to each rotation the 2π phase increase:

$$\phi_M = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n, \qquad t_n \le t < t_{n+1}, \qquad (1)$$

where t_n is the time of the n-th crossing of the secant surface. Note that also in the case of periodic oscillations this definition yields the correct phase. Defined in this way, the phase is a piecewise-linear function of time. It is clear that shifts of this phase do not grow or decay in time, so it corresponds to the direction with the zero Lyapunov exponent. However, this phase crucially depends on the choice of the Poincaré map, and therefore it may be ambiguous.

A.2 Based on a phase space projection

If the above mentioned projection is found, we can also introduce the phase as the angle between the projection of the phase point on the plane and a given direction on the plane (see also [21], [22]):

$$\phi_P = \arctan(y/x) \; . \qquad (2)$$

Note that although the two phases ϕ_M and ϕ_P do not coincide microscopically, i.e on a time scale less than the characteristic period of oscillation, they have equal average growth rates. In other words, the mean frequency defined as the average of $d\phi_P/dt$ over a large period of time coincides with a straightforward definition of the mean frequency via the average number of crossings of a Poincaré surface per unit time.

A.3 Based on the analytic signal

A different way to define the phase is known in signal processing as the analytic signal concept [23]. This general approach, based on the Hilbert transform and originally introduced by Gabor [24], unambiguously gives the instantaneous phase ϕ_H and amplitude for an arbitrary scalar signal $s(t)$. The analytic signal $\zeta(t)$ is a complex function of time defined as

$$\zeta(t) = s(t) + j\tilde{s}(t) = A(t)e^{j\phi_H(t)} \qquad (3)$$

where the function

$$\tilde{s}(t) = \pi^{-1}\text{P.V.} \int_{-\infty}^{\infty} \frac{s(\tau)}{t-\tau} d\tau \qquad (4)$$

is the Hilbert transform of $s(t)$ and P.V. means that the integral is taken in the sense of the Cauchy principal value. The instantaneous amplitude $A(t)$ and the instantaneous phase $\phi_H(t)$ of the signal $s(t)$ are thus uniquely defined from (3). Although the analytic signal approach provides the unique definition of the phase of *a signal*, we cannot avoid ambiguity in defining the phase for a *dynamical system*, because the result depends on the choice of the observable. Here we face the same problem as in the choice of the appropriate projection used for definition of ϕ_M and ϕ_P. However, one can often find an "oscillatory" observable that provides the Hilbert phase ϕ_H in agreement with our intuition.

An important advantage of the analytic signal approach is that the phase can be easily obtained from experimentally measured scalar time series, or in other situations when the construction of the Poincaré map is difficult.

B. Dynamics of the phase of chaotic oscillations

In contrast to the dynamics of the phase of periodic oscillations, the growth of the phase in the chaotic case cannot generally be expected to be uniform. Instead, the instantaneous frequency depends in general on the amplitude, so one can write [25]

$$\frac{d\phi}{dt} = \omega + F(A) \; . \qquad (5)$$

The term $F(A)$ describes the dependence of the instantaneous frequency on the amplitude $A(t)$, which we assume to be chaotic. Note that we use here and below the term "amplitude" in a rather wide sense, referring to all "not-phase" variables (e.g., the variables on the Poincaré surface of section). This chaotic force $F(A)$ can be considered as some effective noise and Eq. (5) is similar to the equation describing the evolution of phase of a periodic oscillator in the presence of external noise. Thus, the dynamics of the phase is generally diffusive: for large t one expects

$$< (\phi(t) - \phi(0) - \omega t)^2 > \sim D_p t,$$

where the diffusion constant D_p measures the strength of the effective noise and quantifies the phase coherence of the chaotic oscillations. It is a particular characteristic of chaotic oscillations which does not coincide with usual ones, e.g. the Lyapunov exponents; it does not exist for general discrete-time dynamical systems.

Generalizing Eq. (5) in the spirit of the theory of periodic oscillations to the case of periodic external force, we get

$$\frac{d\phi}{dt} = \omega + G(\phi, \psi) + F(A) \; , \qquad \frac{d\psi}{dt} = \nu \; , \qquad (6)$$

where G is 2π-periodic in both arguments. This equation is similar to the equation describing synchronization of noisy periodic oscillators. Thus, we expect that in general the synchronization phenomena for periodically forced chaotic systems are similar to those in noisy driven periodic oscillations. One should be aware, however, that the "noisy" term $F(A)$ can be hardly explicitly calculated, and, for sure, cannot be considered as a Gaussian δ-correlated noise as is usual in a statistical approach.

C. Phase of chaotic oscillations: examples

In this subsection we introduce two chaotic systems that serve as prototype models for the subsequent study, and discuss their phase coherence properties.

C.1 KPR circuit.

As the first model we take a simple chaotic generator with the tunnel diode in the LC-loop, it has been introduced and studied by Kijashko, Pikovsky, and Rabinovich [26], [27], [28]. The circuit is shown in Fig. 1, and the equations of motion of externally driven circuit are

$$\begin{aligned} LC\dot{I} &= MS(I-J)+C(U-V)\,, \\ C\dot{U} &= J-I\,, \\ \tilde{C}\dot{V} &= I-I_{td}(V)\,. \end{aligned} \tag{7}$$

The amplifier is assumed to be linear, with the characteristics $I_{out} = SU_{in}$, and the only nonlinear element in the circuit is the tunnel diode, whose current–voltage characteristics $I_{td}(V)$ has a usual N-shaped form. In dimensionless variables $x \sim I$, $y \sim U$, $z \sim V$ the equations are:

$$\begin{aligned} \dot{x} &= h(x-E\cos\nu t)+y-z\,, \\ \dot{y} &= -x+E\cos\nu t\,, \\ \mu\dot{z} &= x-f(z)\,. \end{aligned} \tag{8}$$

where $h = MS(LC)^{-1/2}$ and $\mu = \tilde{C}/C$ are parameters and the dimensionless tunnel diode characteristic $f(z)$ is approximated by $f(z) = -z + 0.002\sinh(5z-7.5)+2.9$. The parameter E is proportional to the amplitude of the driving current $J \sim E\cos\nu t$.

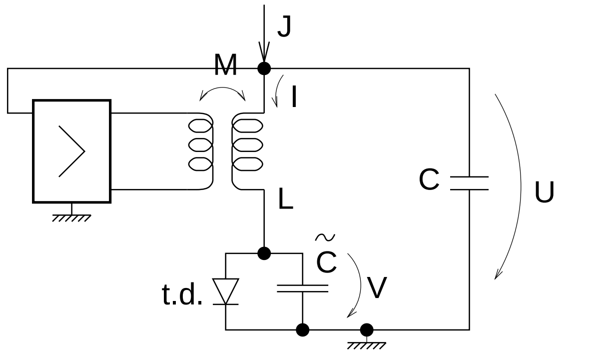

Fig. 1. Scheme of the KPR circuit driven by external current J.

In Fig. 2a we present the phase portrait of the autonomous ($E = 0$) KPR generator. The phase diffusion coefficient is $D_p = 0.0013$, i.e. the effective noise is low and the phase coherence is high. It corresponds to a sharp peak in the power spectrum (Fig. 2a). Because the phase portrait is topologically simple, all definitions of the phase give practically identical results.

C.2 Rössler system.

As the second model we take the well–studied Rössler system with external periodic driving [29]

$$\begin{aligned} \dot{x} &= -y-z+E\cos\nu t\,, \\ \dot{y} &= x+ey\,, \\ \dot{z} &= f+z(x-c)\,, \end{aligned} \tag{9}$$

where e, f and c are parameters. In this section we discuss only the autonomous case $E = 0$.

For parameter values $e = 0.15$, $f = 0.4$ and $c = 8.5$ this attractor has a sharp peak in the power spectrum and a rather simple form (similar to Fig. 2a,b). Here the Poincaré map can be easily constructed, and all three definitions of the phase also give similar results. The diffusion constant for this attractor is extremely small ($D_p < 10^{-4}$) comparing to the KPR circuit. This small diffusion corresponds to an extremely sharp peak in the spectrum (see also discussion in [30], [31], [32]). Thus, this attractor can be called phase-coherent.

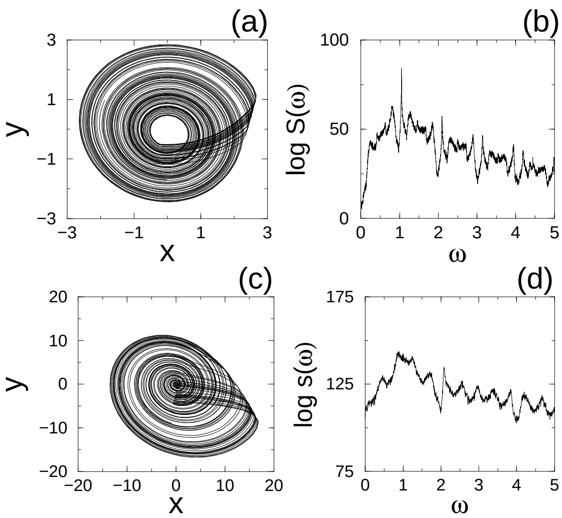

Fig. 2. The phase portrait and the power spectrum of $x(t)$ for the KPR oscillator ((a) and (b)) and for the funnel Rössler attractor ((c) and (d)).

For the parameter set $e = 0.25$, $f = 0.4$ and $c = 8.5$ the Rössler system demonstrates the so-called funnel attractor, shown in Fig. 2c. The topological structure is now more complex: there are small and large loops on the x, y plane, and it is not clear which phase shift (π or 2π) should be attributed to these loops. Consequently, different definitions of the phase give different results. The effective noise in the funnel Rössler attractor is extremely large: different approaches to the phase definition give D_p about 0.3 (cf. Fig. 2d).

III. Phase and frequency locking of chaotic oscillators by an external force

If the phase of a chaotic oscillator is well-defined, i.e. all approaches to the definition of the phase give similar results, we can use the coincidence of the observed frequencies as the criterion of synchronization. We emphasize that the mean frequency of chaotic oscillations Ω can be calculated rather easily: as it follows from (1)

$$\Omega = \lim_{t\to\infty} 2\pi\frac{N_t}{t} \tag{10}$$

where N_t is the number of crossings of the Poincaré section during observation time t. This method can be straightforwardly applied to observed time series; in the simplest case one can, e.g., take for N_t the number of maxima of $x(t)$.

A. *Direct phase calculation*

Here we demonstrate synchronization of the KPR circuit (8). The mean frequency was calculated using (10). The dependence of $\Delta\Omega = \Omega - \nu$ on the amplitude E and the frequency of the external force ν (Fig. 3) exhibits clearly that there exist such phase-locking regions which correspond to the main resonance $\nu \approx \omega_0$ (Fig. 3a) and to the resonances $\nu \approx 2\omega_0$ (Fig. 3b) and $2\nu \approx \omega_0$ (Fig. 3c). While the main synchronization region is rather large and sets in already for very small amplitudes, higher resonances are weaker and can be observed only for larger forcing.

When simulating the forced Rössler system (9), we have obtained results very similar to these described above, but synchronization appears practically without threshold. We attribute this to the extremely high phase coherence of the Rössler attractor. For the funnel Rössler attractor the phase is ill-defined, so we need special tools to characterize the phase synchronization indirectly.

B. *Indirect characteristic of phase synchronization*

It would be useful to have characteristics of synchronization which do not depend on the definition of the phase and where it is not necessary to compute the phase explicitly [25], [33]. If the phase of a chaotic system is locked by a periodic force, the process becomes highly correlated in time: the values of an observable u at times t and $t + nT$ (we remind that T is the period of the external force) differ only due to the chaotic nature of the amplitudes, because the phases at these times are almost identical. This can be seen by calculating the autocorrelation function $C(\tau) = < u(t)u(t+\tau) >$ which has a periodic tail for $\tau \to \infty$ with maxima at $\tau = nT$. Thus, in the power spectrum high δ-peaks appear at the frequency of external force Ω and its harmonics $n\Omega$. Therefore, one can characterize the phase synchronization by calculating the discrete part of the power spectrum. An appropriate quantity is the intensity of the discrete spectrum defined according to the Wiener lemma [34] as

$$S = \lim_{t\to\infty} \frac{1}{t} \int_0^t C^2(\tau)\, d\tau \, . \tag{11}$$

A resonance-like curve S vs Ω is an indicator of the phase synchronization. Other indirect characteristics are discussed in [33].

We have applied the approach described to the periodically forced funnel Rössler system (4). One can see that there exists a range of external frequencies where the intensity of the discrete spectral component has a maximum, although this maximum is much smaller than for the phase-coherent Rössler attractor. We suggest to interpret these results as a manifestation of the phase synchronization phenomenon in a system with very large effective noise (the noise in the phase dynamics comes from the switching between large and small loops).

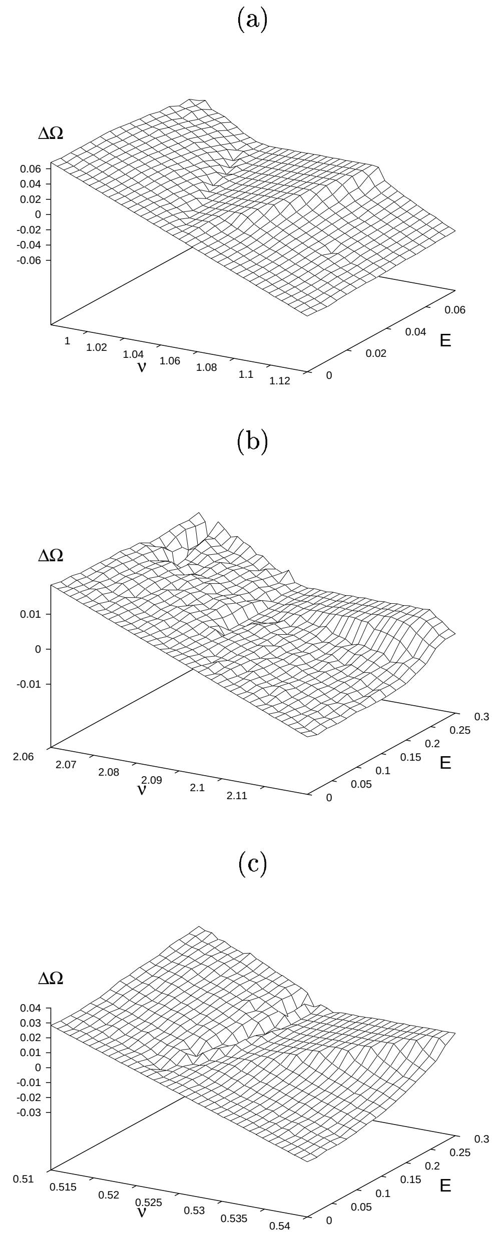

Fig. 3. The "Arnold tongues" of the forced KPR generator (8) for $h = 0.2$, $\mu = 0.1$. The main resonance and the resonances 1:2 and 2:1 are shown.

IV. Interacting chaotic oscillations

In this section we describe *mutual* phase synchronization in coupled chaotic oscillators.

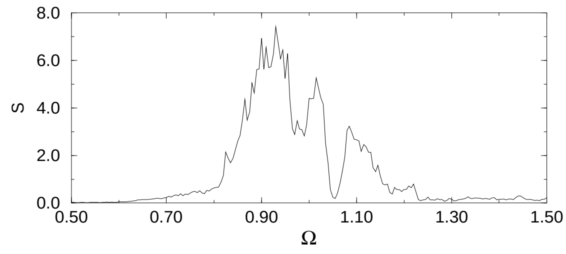

Fig. 4. Indirect manifestation of phase synchronization: the intensity of the discrete component of power spectrum for the funnel Rössler oscillator is plotted vs. driving frequency Ω for $E = 0.5$.

A. Transition to phase synchronization

We start with a system of two coupled Rössler oscillators [35]:

$$\begin{aligned} \dot{x}_{1,2} &= -\omega_{1,2} y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} &= \omega_{1,2} x_{1,2} + e y_{1,2}, \\ \dot{z}_{1,2} &= f + z_{1,2}(x_{1,2} - c), \end{aligned} \tag{12}$$

The oscillators are not identical, i.e. $\omega_{1,2} = 1 \pm \Delta$.

As the coupling is increased for a fixed mismatch Δ, we observe a transition from a regime, where the phases rotate with different velocities $\phi_1 - \phi_2 \sim \Delta \cdot t$, to a synchronous state, where the phase difference does not grow with time, i.e. $|\phi_1 - \phi_2| <$ const, and $\Delta\Omega =< \dot{\phi}_1 - \dot{\phi}_2 >= 0$ (Fig. 5). We emphasize that in contrast to the other types of synchronization of chaotic systems [6], [7], [8], [36], [37], here the instant vectors $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ do not coincide. Moreover, the correlations between the amplitudes $A_{1,2} = \sqrt{x_{1,2}^2 + y_{1,2}^2}$ are pretty small, although the phases are completely locked.

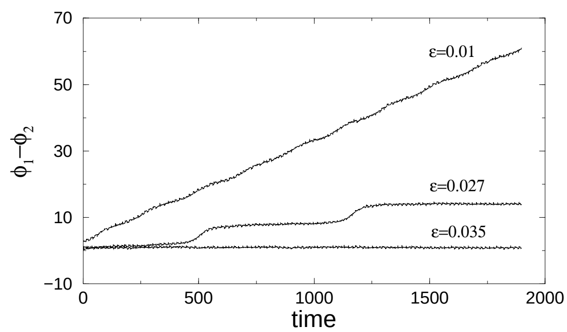

Fig. 5. Phase difference of two coupled Rössler systems (eq. (12)) vs. time for non–synchronous, ($\varepsilon = 0.01$), nearly synchronous, or intermittent, ($\varepsilon = 0.027$) and synchronous ($\varepsilon = 0.035$) states (a). In the last case the amplitudes $A_{1,2}$ remain chaotic (b), their cross-correlation is less than 0.2. The frequency mismatch is $\Delta = 0.015$. Parameter values: $e = 0.15$, $f = 0.2$, $c = 10$.

An interesting feature is the appearance of intermittency at the onset of synchronization. Indeed, as one can see from Fig. 5a, at the border of the region of complete phase locking, the phases are almost locked. It means that from time to time phase slips occur, where during a rather small interval of time the phase difference changes by 2π. The time intervals between these slips are irregular, as one can see from their distribution (Fig. 6). The slips are exponentially rare, and the dependence of the number of phase slips per constant time N_s on the coupling strength obeys a relation $N_s \sim \exp(-|\varepsilon - \varepsilon_c|^{-1/2})$ [38] (Fig. 6b).

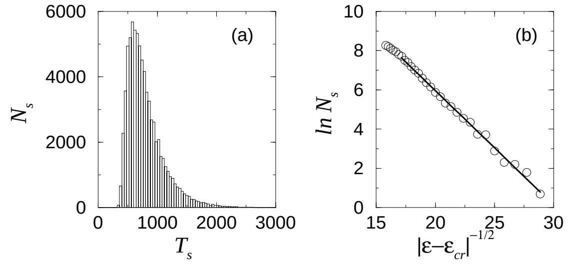

Fig. 6. (a): The distribution of the number of phase slips N_s with the interval between slips T_s for $\varepsilon = 0.027$; it demonstrates that the slips occur irregularly. (b) The number of phase slips per constant time N_s vs. the coupling strength in the vicinity of the transition point. The slips are exponentially rare.

B. Lyapunov exponents

Here we consider two KPR circuits (Fig. 1) coupled by a resistor connecting the inputs of amplifiers:

$$\begin{aligned} \dot{x}_{1,2} &= \omega_{1,2}^2 [h(x - \varepsilon(y_{2,1} - y_{1,2})) + y_{1,2} - z_{1,2}] \ , \\ \dot{y}_{1,2} &= -x_{1,2} + \varepsilon(y_{2,1} - y_{1,2}) \ , \\ \mu \dot{z}_{1,2} &= x_{1,2} - f(z_{1,2}) \ . \end{aligned} \tag{13}$$

For simplicity, we assume that in both circuits only the inductances are different.

In order to describe the phase synchronization transition in the framework of transitions in chaotic systems, we have studied the Lyapunov exponents. In Fig. 7 we present the 4 largest Lyapunov exponents for the system (13) in dependence on the coupling strength ε. In the uncoupled case each oscillator has one positive, one zero, and one negative Lyapunov exponent, the zero ones corresponding to the phases. For $\varepsilon < 0.04$ the phases are not locked, and two nearly zero Lyapunov exponents are observed. We see from Fig. 7 that the transition to phase synchronization happens, when one of these zero Lyapunov exponents becomes negative, corresponding to a stable relation between the phases (one Lyapunov exponent is exactly zero, it corresponds to a simultaneous shift of both phases). The frequency difference vanishes, however, not exactly at the point where the Lyapunov exponent becomes negative, because of relatively large effective noise of the phase motion caused by chaotic amplitudes. Note also that at the phase synchronization transition there are two positive Lyapunov exponents corresponding to the amplitudes. Thus this transition can be characterized as a transition inside chaos (to be more precise, inside hyperchaos) and not as a "chaos–order" transition.

V. Applications to data analysis

The analysis of phase relationships between two signals, naturally arising in the context of phase synchronization,

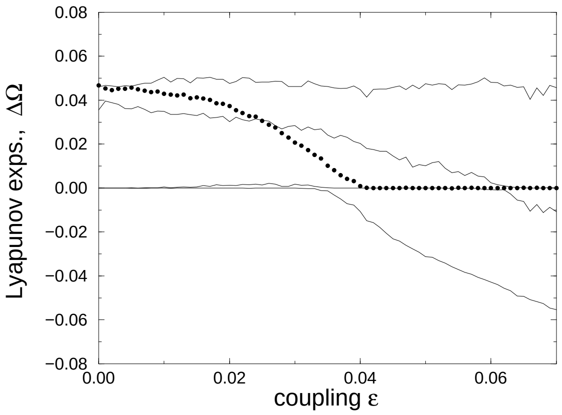

Fig. 7. The four largest Lyapunov exponents (lines) and the frequency difference (closed circles) for the coupled KPR circuits (13); $\omega_1 = 0.98$, $\omega_2 = 1.02$, $h = 0.2$, $\mu = 0.1$. The phase synchronization transition is observed at $\varepsilon \approx 0.04$.

can be used to approach a general problem in time series analysis. Namely, bivariate data are often encountered in the study of real systems, and the usual aim of the analysis of these data is to find out whether two signals are dependent or not. As experimental data are very often nonstationary, traditional techniques, such as cross–spectrum and cross–correlation analysis [23], or non–linear characteristics like generalized mutual information [39], [40] have their limitations. From the other side, sometimes it is reasonable to assume that the observed data originate from two weakly interacting systems with slowly varying parameters. If the signals are close to periodic ones, the usual approach is to consider them as an output of two coupled oscillators and to quantify their interaction by measuring the time dependent phase difference between these signals. Here we demonstrate that this approach can be extended to the case of chaotic signals as well. In this case the phase difference can be effectively obtained from a bivariate time series by means of the analytic signal approach based on the Hilbert transform [23]. An important advantage of the analytic signal approach is that the phase can be easily obtained from experimentally measured scalar time series [41].

The real word is often (if not always) non–stationary. Parameters of interacting subsystems and/or of coupling typically vary with time. Nevertheless, as the stationarity of the time series is not required for the Hilbert transform, we can calculate the phase difference and find epochs of synchronous and non-synchronous behavior.

To illustrate this, we present the result of experiments on posture control in neurological patients [42]. During these tests a patient is asked to stay quiet on a special rigid force plate with four tensoelectric transducers. The output of the setup provides current coordinates (x, y) of the center of pressure under the feet of the standing subject. These bivariate data are called stabilograms; they are known to contain rich information on the state of the central nervous system. In the following we denote the deviation of the center of pressure in anterior–posterior and lateral direction as x and y, respectively. Every subject was asked to perform three tests of quiet standing with (a) eyes opened and stationary visual surrounding (EO); (b) eyes closed (EC); (c) eyes opened and additional video–feedback (AF). In order to eliminate low–frequency trends, a moving average computed over the n–point window was subtracted from the original data. The window length n has been chosen by trial to be equal or slightly larger than the characteristic oscillation period. Its variation up to two times does not practically effect the results.

Here we present in detail the results of the analysis of one trial (female subject, 39 years old, functional ataxia). We can see that in the EO and EC tests the stabilograms are clearly oscillatory (Fig. 8). The difference between these two records is that with eyes opened the oscillations in two directions are not synchronous during approximately the first 110s, but are phase locked during the last 50s. In the EC test, the phases of oscillations are perfectly entrained during all the time. The behavior is essentially different in the AF test; here no phase locking is observed. It is noteworthy that conventional methods of time series analysis, such as cross–spectrum analysis or characteristic of non–linear relationship like the generalized mutual information function are not efficient in this case and does not reveal only significant dependence even in the EC test, where the phases are completely locked. It is important to emphasize that synchronous epochs have been found in the cases of pathology only [42].

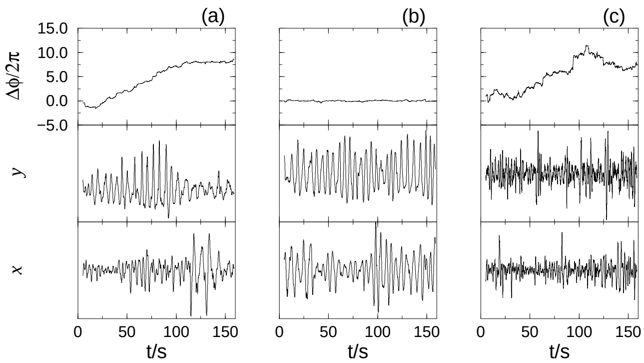

Fig. 8. Stabilograms of an neurological patient for EO (a), EC (b), and AF (c) tests. The upper panels show the relative phase between two signals x and y. During the last 50s of the first test and the whole second test the phases are perfectly locked. No phase entrainment is observed in the AF test.

VI. Conclusions

The main idea of this paper is to develop a unified framework for the description of synchronization effects both in periodic and chaotic oscillators. We achieve it by extending the notion of phase to the case of chaotic systems. Although we are not able to do this rigorously and to suggest a unique definition of the phase, we have shown that it can be introduced in some reasonable and consistent way for different types of chaotic oscillators. We have proposed and compared three approaches to the definition of the phase.

The dynamics of the phase of a chaotic system is similar to that of a periodic oscillator in the presence of noise. Here, the chaotic behavior of amplitudes acts as some effective noise, although it is of purely deterministic origin. Because of this similarity in phase dynamics, we expect that many, if not all, synchronization features known for periodic oscillators can be observed for chaotic systems as well.

Indeed, we have demonstrated effects of phase and frequency entrainment by periodic external driving or due to interaction of two chaotic oscillators. In the synchronous regime, the phases or frequencies of the oscillators are entrained, while the amplitudes of the oscillator(s) remain chaotic. This effect appears to be robust enough to be observed in physical experiment [25], [43] and in living nature. Noteworthy, even in the case when the phases are not well-defined, the presence of phase synchronization can be demonstrated indirectly, i.e. independently of any particular definition of the phase.

Among other phase synchronization phenomena already studied for chaotic oscillators we refer to the effect of phase synchronization in a lattice of chaotic oscillators [44] and self-synchronization in a large ensemble of non-identical globally coupled oscillators [21]. The latter effect manifests itself as an appearance. Being observed both for oscillators with different phase coherence properties, this effect is caused by mutual phase entrainment.

Finally, we would like to stress that contrary to other types of chaotic synchronization, phase synchronization phenomena can happen already for very small coupling, which offers an easy way of chaos regulation.

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