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# Symmetry Breaking in Distributed Systems and Modulational Spatio–Temporal Intermittency

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**Abstract**—It is shown that at the symmetry-breaking transition of spatio–temporal chaos a new type of spatio–temporal intermittency is observed. This regime is a direct analogue of modulational intermittency previously investigated in nondistributed systems. Statistical properties of modulational spatio–temporal intermittency are investigated, and correspondence to the Kardar–Parisi–Zhang equation is established.

Chaotic behaviour in distributed dynamical systems (spatio–temporal chaos) is intensively investigated now. It can be observed in different types of models: partial differential equations, differential-delay equations, coupled map lattices, as well as in numerous experiments. Spatio–temporal chaos is characterized by excitation of a large number of degrees-of-freedom; spatial and temporal correlations usually decay exponentially in space and time; distribution of Lyapunov exponents approaches for large systems a thermodynamic limit [1]. Many features of spatio–temporal chaos can be understood from the dynamics of low-dimensional systems, e.g. period-doubling route to chaos in distributed systems is described by a straightforward generalization of a renormalization group for one-dimensional maps [2]. Some other features appear only in distributed systems, e.g., the spatio–temporal intermittency [3], for which laminar and turbulent regions coexist in space, is similar to percolation phenomenon and has no direct analogy in non-distributed case.

In this paper we consider a new type of spatio–temporal intermittency, which appears not at the transition to chaos, but within spatio–temporal chaos. It is a direct analogue to modulational intermittency in non-distributed systems, studied in refs [4, 5]. We remind now the properties of modulational intermittency. Physically, it appears when the whole system can be divided into two parts: one part (subsystem) behaves chaotically, and this chaotic motion modulates another, initially nonchaotic field. In the simplest case the mechanism of modulational intermittency can be illustrated with the following simple example

$$\begin{aligned} s_{t+1} &= f(s_t) \\ p_{t+1} &= \phi(s_t)p_t. \end{aligned} \tag{1}$$

Here the field $s$ is governed by a one-dimensional mapping with chaotic dynamics, and the field $p$ is modulated by it. In fact, for $p$ we have multiplicative noise excitation, which gives intermittent behaviour [6, 7].

System (1) arises naturally when the symmetry-breaking transition in chaos is investigated [8, 5]. Let us consider two coupled one-dimensional mappings with chaotic behaviour

$$\begin{aligned} u_{t+1} &= (1-\sigma)f(u_t) + \sigma f(v_t), \\ v_{t+1} &= \sigma f(u_t) + (1-\sigma)f(v_t). \end{aligned} \tag{2}$$

For any value of coupling constant $d$ there is a symmetric solution $u_t = v_t$, but it loses stability at the critical coupling $\sigma < \sigma_c = (1 - e^{-\lambda})/2$, where $\lambda$ is the Lyapunov exponent of chaotic synchronous oscillations. Near the threshold, introducing the variables

$$s = \frac{u + v}{2}, \qquad p = \frac{u - v}{2}$$

we obtain in the first order in $p$, exactly the system (1) with $\phi(s) = f'(s)(1 - 2\sigma)$. Thus, near the symmetry-breaking transition the modulational intermittency is observed [8, 5, 7]. Similar regimes may also occur at the other cases of synchronization transition in chaos [9, 10].

We now generalize the construction above to the case of spatio–temporal chaos. As an elementary model we take a coupled map lattice (CML)

$$u(x, t + 1) = f(\hat{D}u(x, t)). \tag{3}$$

Here the field $u$ depends on discrete time $t$ and discrete spatial coordinate $x$, $f$ is a nonlinear function, and $\hat{D}$ is a diffusion operator:

$$\hat{D}u(x) = \varepsilon u(x - 1) + (1 - 2\varepsilon)u(x) + \varepsilon u(x + 1).$$

The system (3) is a standard model in studies of spatio–temporal dynamics [11–13]. We will consider the case, when (3) demonstrates chaotic behaviour with decreasing space and time correlation functions. (Boundary conditions are throughout this paper periodic: $u(x + L) = u(x)$.)

Let us consider two coupled CMLs:

$$\begin{aligned} u(x, t + 1) &= (1 - \sigma)f(\hat{D}u(x, t)) + \sigma f(\hat{D}v(x, t)) \\ v(x, t + 1) &= \sigma f(\hat{D}u(x, t)) + (1 - \sigma)f(\hat{D}v(x, t)). \end{aligned} \tag{4}$$

To this system we can apply the same approach as to (2), and conclude that symmetric solution loses stability at $\sigma_c = (1 - e^{-\lambda})/2$, where now $\lambda$ is the maximal Lyapunov exponent for spatio–temporal chaos. Near the threshold a regime is observed, which we call spatio–temporal modulational intermittency (Fig. 1). On a snapshot of the difference field $p = (u - v)/2$ one can see laminar regions interrupted with turbulent bursts; a similar picture appears when a field in one site is drawn as a function of time. The global behaviour of turbulent bursts is presented at Fig. 2. Here we do not see the characteristic for usual spatio–temporal intermittency, regular behaviour of laminar–turbulent interfaces [14]. The physical mechanism for spatio–temporal modulation is as follows. Evolution of the difference field can be written in the linear approximation as

$$p(x, t + 1) = (1 - 2d)f'(\hat{D}(\varepsilon)s(x, t))\hat{D}(\varepsilon)p(x, t). \tag{5}$$

Where $s = (u + v)/2$ obeys (3). We see that the field $p$ is locally multiplied by chaotic factors. In the sites where these factors were occasionaly sufficiently large for some time, large bursts are observed, and due to diffusion these bursts have finite size. When period of large factors ends, large bursts are observed in another places.

We present now another example of modulational intermittency in a system with infinite number of degrees of freedom – in coupled differential-delay equations

$$\begin{aligned} \frac{du}{dt} &= f(u(t - \tau)) + \sigma(v(t) - u(t)), \\ \frac{dv}{dt} &= f(v(t - \tau)) + \sigma(u(t) - v(t)). \end{aligned} \tag{6}$$

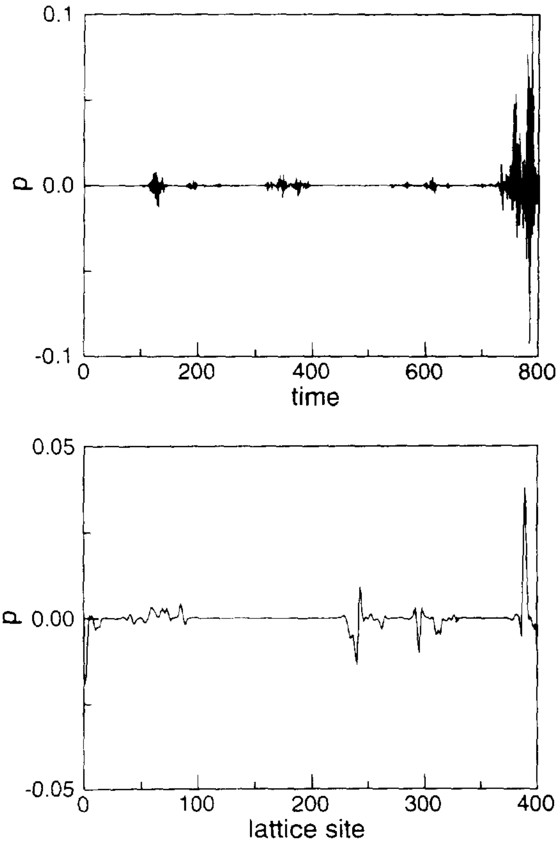


Fig. 1. Modulational spatio-temporal intermittency in coupled CMLs with logistic maps $f(u) = 4u(1-u)$ for $\varepsilon = 1/3$ and $\sigma = 0.155$. (a) Difference field $p$ as a function of time at one site. (b) Snapshot of the difference field.

Here time and 'space' are not well separated, and therefore the threshold of synchronization $\sigma_c$ cannot be simply related to the largest Lyapunov exponent of the uncoupled system. Instead, one has to solve the linearized equation for $p = (u - v)/2$

$$\frac{dp}{dt} = f'(s(t-\tau))p(t-\tau) - \sigma p(t)$$

and to find from it the effective Lyapunov exponent $\lambda(\sigma)$. The threshold coupling $\sigma_c$ is obtained from the condition $\lambda(\sigma_c) = 0$. Apart from these differences, a regime observed near the threshold (Fig. 3) is completely similar to that in coupled CMLs.

In the rest of the paper we present a theory of spatio-temporal modulational intermittency for one particular case, when the field governed by equation (5) is strictly positive. For this end we need the derivative $f'$ in (5) to be positive. So below we consider a piecewise-linear map $f$ having everywhere a positive slope.

Our goal is to study the statistical properties of the perturbation field $p$ for large system size $L$ and time $t$. We show that in this case the dynamics of (5) may be described by the Kardar–Parizi–Zhang (KPZ) equation, derived previously for growing interfaces in a random medium [15]. Indeed, equation (5) may be considered as a discrete analogue of the diffusion equation with multiplicative noise

$$\frac{\partial W}{\partial t} = \xi(x,t)W + R\frac{\partial^2 W}{\partial x^2}. \tag{7}$$

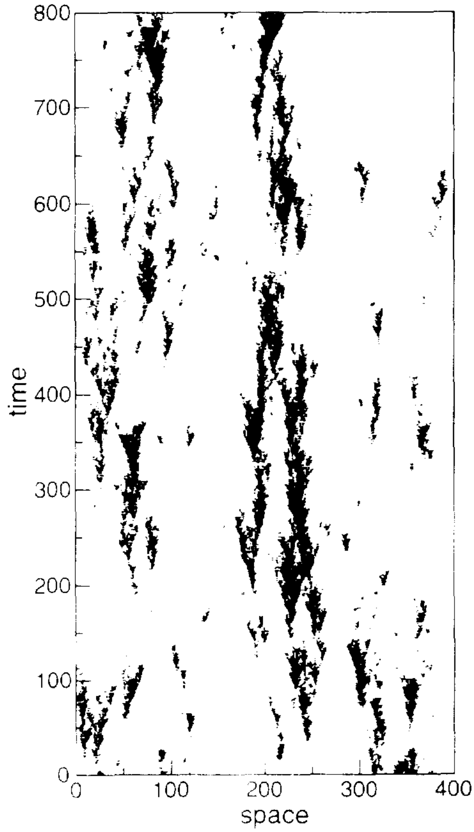


Fig. 2. Spatio-temporal dynamics of the regime presented at Fig. 1. Sites where $|p| > 0.01$ are marked with dark squares.

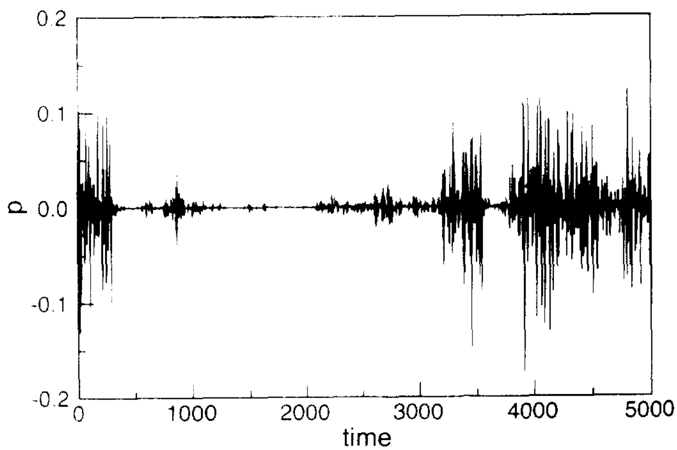


Fig. 3. Modulational intermittency in coupled differential-delay equations (6) for $f(u) = 4u(1-u)$, $\tau = 5$, $\sigma = 0.2$.

This equation with the ansatz $W = \exp(H)$ is transformed to the KPZ equation [15]

$$\frac{\partial H}{\partial t} = \frac{\lambda}{2}\left(\frac{\partial H}{\partial x}\right)^2 + \nu\frac{\partial^2 H}{\partial x^2} + \xi(x, t). \tag{8}$$

This equation describes kinetic roughening of randomly-driven interfaces and has been thoroughly investigated in recent years [16]. If the KPZ equation is derived from equation (7), one has $\lambda = 2R$, $\nu = R$. In the standard KPZ equation it is assumed that the noise $\xi(x, t)$ is Gaussian and $\delta$-correlated:

$$\langle \xi(x, t)\xi(x', t')\rangle = D\delta(x - x')\delta(t - t').$$

We now explore the analogy between the discrete equation (5) and the multiplicative noise equation (7) and apply the ansatz

$$p(x, t) = e^{h(x,t)}. \tag{9}$$

Then we get from (5) a discrete analogue of the KPZ equation:

$$\begin{aligned} h(x, t+1) - h(x, t) &= \ln a(x, t) \\ &\quad + \ln[1 - 2\varepsilon + \varepsilon \exp(h(x-1, t) - h(x, t)) + \varepsilon \exp(h(x+1, t) - h(x, t))]. \end{aligned} \tag{10}$$

Here $a(x, t) = (1 - 2d)f'(\hat{D}(\varepsilon)s(x, t))$. It is worth noting that for the discrete case there is an important restriction in performing the ansatz (9), namely, $p(x, t)$ should be positive for all $x$, $t$. In the continuous case this can be ensured by a proper choice of the initial field, while in the discrete case also the condition $a(x, t) > 0$ must be fulfilled for all $x$, $t$.

It follows from (9) that the exponential growth of the field $w(x, t)$ in time corresponds to the linear motion of the interface position $h(x, t)$; the mean velocity is exactly the Lyapunov exponent. Except for this mean motion, the interface $h(x, t)$ also fluctuates (due to fluctuations of $a(x, t)$) and we now can investigate these fluctuations using the correspondence to the KPZ equation.

Because $\varepsilon$ is an effective diffusion constant corresponding to $R$ in equation (7), (10) corresponds to the KPZ equation (8) with

$$\lambda = 2\varepsilon, \qquad \nu = \varepsilon.$$

Note that the parameter $\varepsilon$ is the diffusion constant both in the KPZ equation and in the discrete equation (5). The parameter $\lambda$ in the KPZ equation describes the change in the growth rate of the tilted interface. For the discrete equation (5) this corresponds, because of the ansatz (9), to the change of the Lyapunov exponent when exponentially growing in space perturbations are considered; such generalized Lyapunov exponents have been introduced recently by Politi and Torcini [17]. The problem remains in finding a value for the noise strength $D$. The values of $a(x, t)$ are produced by chaotic motions in the CML (3) and of course are neither Gaussian nor $\delta$-correlated. These differences are, however, not important if the asymptotic behaviour coincides with that predicted by the KPZ equation. While a large number of models belong to the universality class of KPZ equation, we have to check this for the perturbation field in CML once more.

We used in the numerical calculations the following 'skewed' doubling transformation

$$f(u) = \begin{cases} bu & \text{for } 0 \leq u < b^{-1} \\ (b/b-1)u & \text{for } b^{-1} \leq u \leq 1 \end{cases} \tag{11}$$

In this transformation the local instantaneous expansion rate $a(x, t)$ takes the values $b$ and $b(b-1)^{-1}$, so varying the parameter $b$ we can consider both cases of weak ($b \approx 2$) and strong ($b \gg 1$) noise. Numerical simulation shows that the CML (3), (5), (11) indeed demonstrates properties of the KPZ equation. If a system of finite length $L$ is considered, then for sufficiently large $t$ a statistically stationary roughened interface appears (Fig. 4). The probability distribution density of $h$ obeys Gaussian law (Fig. 5), and the spatial spectrum scales as $k^{-2}$, as is expected for the KPZ equation [18] (Fig. 6). From the

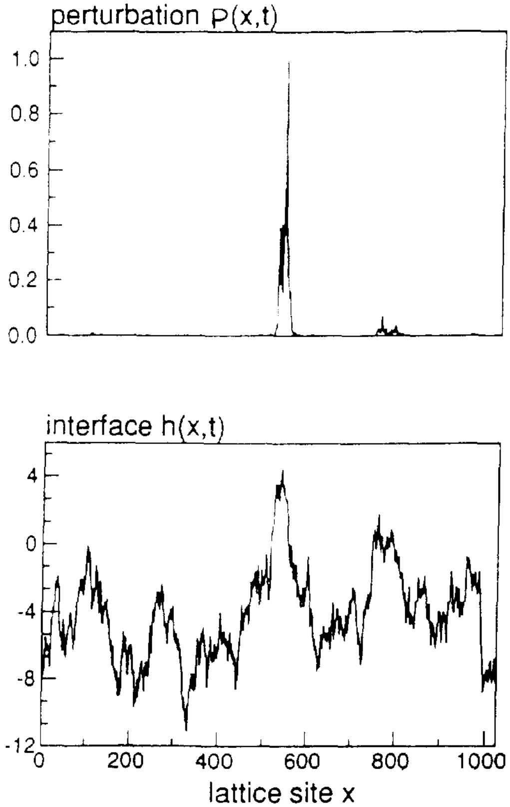


Fig. 4. Snapshot of the fields $p(x, t)$ and $h(x, t)$ for the CML equations (3), (5), (11) with $L = 1024$, $\varepsilon = 0.1$, $a = 4$

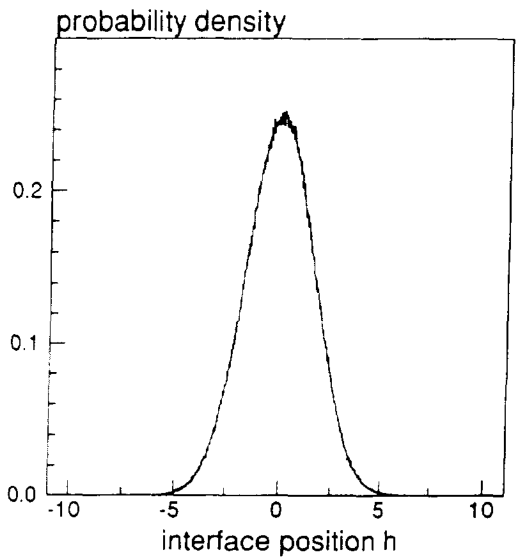


Fig. 5. Probability distribution density of the field $p(x, t)$ for the CML with $\varepsilon = 0.3$, $b = 4$, $L = 256$. The curve nearly coincides with the Gaussian one.

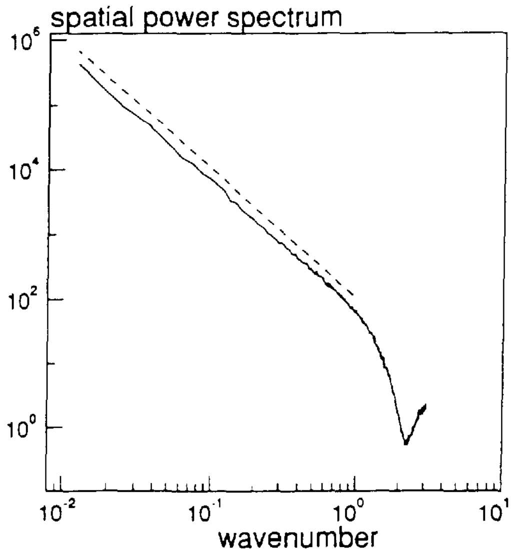


Fig. 6. Spatial spectrum of the field $h(x, t)$. The broken line has slope $-2$.

asymptotic behaviour of system (3), (5) we can also estimate the effective noise strength (this procedure has been recently applied to the Kuramoto–Sivashinsky equation [19]), see ref. [20].

In the terms of the KPZ equation, the spatio-temporal modulational intermittency corresponds to the fact, that the observed field is an exponent of the interface (Fig. 4). Thus only a small region near the maximum of the interface is observed as an isolated burst. This isolation agrees with predicted in ref. [21] localization of eigenfunctions in the linearized equation for spatio-temporal chaos.

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