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Finite-size-induced transition in ensemble of globally coupled oscillators

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Abstract. The collective behavior of overdamped nonlinear noise-driven oscillators coupled via mean field is investigated numerically. When a coupling constant is increased, a transition in the dynamics of the mean field is observed. This transition scales with the number of oscillators and disappears when this number tends to infinity. Analytical arguments explaining the observed scaling are presented.

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The dynamics of large ensembles of coupled oscillators has been intensively investigated now. Usually, two types of coupling are considered: local coupling, when each oscillator is influenced only by its neighbors, and global coupling, when the interaction does not depend on the distance between elements (and in this case it is not important how the oscillators are placed in space). In this paper we consider noise-driven globally coupled oscillators. Systems of this type appear in different fields, including oscillatory neuronal systems [1], multimode lasers [2], Josephson junction arrays [3], etc. Our model is very similar to that introduced by Kometani and Shimizu [4], who considered noise-driven overdamped bistable oscillators, coupled through a mean field. A detailed study of this model was performed by Desay and Zwanzig [5]. They showed that in the thermodynamic limit (number of oscillators N tends to infinity) a phase transition is observed: with decreasing of noise intensity a symmetrical regime with zero mean field becomes unstable, and an asymmetrical state with a non-zero mean field appears. Similar transitions were studied in [6, 7].

In this paper we study a model which has in contrast to the Komitani-Shimizu model a nonlinear dependence of oscillators potential on the mean field. A phase transition, which is observed in this system, has rather unusual properties: it is observed only for finite ensembles and disappears in the thermodynamic limit. Therefore we call this transition a "finite-size-induced". To the best of our knowledge, such transitions have been not reported for globally coupled oscillator systems.

The model we will deal with is an ensemble of N identical noise-driven oscillators. Their dynamics is described by the Langevin equation

$$\frac{dx_i}{dt} = -\frac{\partial U(x_i, S)}{\partial x_i} + \sqrt{2D}\,\xi_i(t), \qquad i=1, \ldots, N. \tag{1}$$

Here $\xi_i(t)$ is δ-correlated Gaussian noise: $\langle \xi_i(t)\,\xi_j(t') \rangle = \delta_{ij}\delta(t-t')$, and U is a potential which is assumed to depend on the mean field S defined as

$$S = \frac{1}{N}\sum_1^N x_i. \tag{2}$$

We assume that the potential linearly depends on the mean field

$$U(x, S) = U_0(x) + \varepsilon S U_1(x) \tag{3}$$

while both U_0 and U_1 may nonlinearly depend on x. Here we have also introduced the coupling constant ε, which will be our main parameter. Note, that the Komitani-Shimizu model corresponds to the case, when U_0 is a bistable potential, while $U_1(x)=x$. In Appendix we describe a set of electronic oscillators, governed by Eq. (1), and discuss under which conditions the coupling potential can be nonlinear. Below we consider the following potentials:

$$U_0(x) = \frac{x^4}{4}, \qquad U_1(x) = -\frac{x^2}{2} \tag{4}$$

so (1) has the form

$$\frac{dx_i}{dt} = -x_i^3 + \varepsilon S x_i + \sqrt{2D}\,\xi_i(t). \tag{5}$$

Physically, uncoupled oscillators are at the border of bistability. A positive mean field leads to a bistable local potential, while for negative mean field small oscillations become linear.

Let us first consider the system in the thermodynamic limit $N \to \infty$. In this limit the self-consistent mean-field approach of Desay and Zwanzig [5] is valid, and the ensemble is described by a nonlinear Fokker-Planck equation for the probability density $W(x, t)$:

$$\frac{\partial W}{\partial t} + \frac{\partial}{\partial x}[(\varepsilon S x - x^3) W] = D \frac{\partial^2 W}{\partial x^2}, \tag{6a}$$

$$S = \int x W(x) \, dx. \tag{6b}$$

It easy to see that the only stationary solution of this equation has a vanishing mean field $S=0$. Indeed, assuming that $S=S_0=\text{const}$, we can solve (6a) to obtain

$$W(x) = W_0 \exp\left(\frac{\varepsilon S x^2}{2} - \frac{x^2}{4}\right), \tag{7}$$

W_0 being a normalization factor. This distribution density is symmetrical, so from (6b) it follows that $S_0=0$. Thus, there is no phase transition in the thermodynamic limit.

We have studied the dynamics of finite ensembles numerically, solving the system of Langevin equations (5), (2) with the 1-step Euler method. For fixed parameters N and D we have observed a transition as the coupling strength ε was increased. Typical behavior of the mean field S is presented in Fig. 1. For $\varepsilon=7$ (slightly below the transition) the mean field fluctuates near zero, although high but rate bursts are already seen. As the

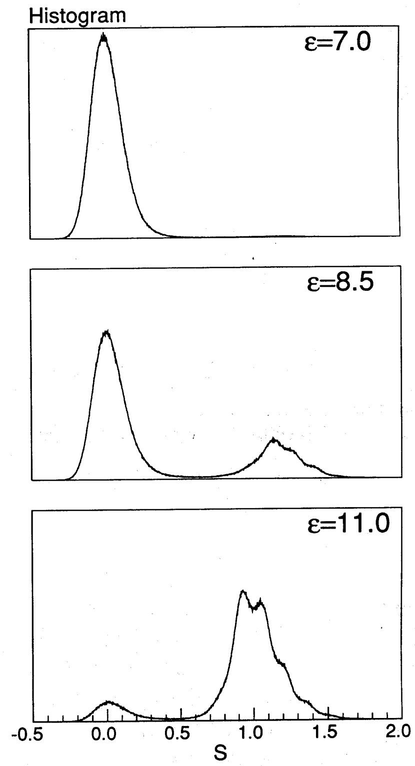

Fig. 2. Histograms of the mean field (in arbitrary units) for the regimes presented in Fig. 1

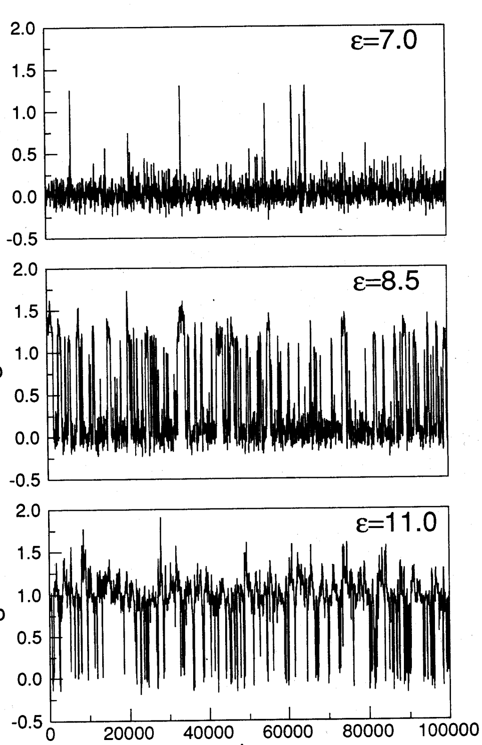

Fig. 1. Dynamics of mean field S in the ensemble (5) for $N=100$, $D=2$

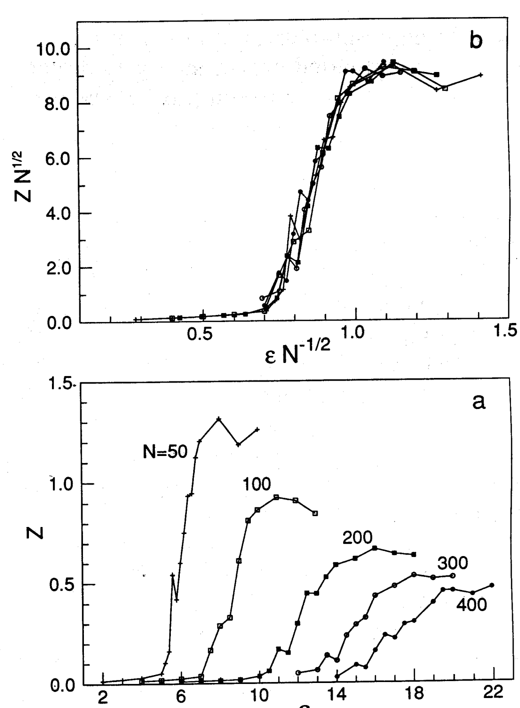

Fig. 3. a Averaged mean field vs. coupling constant for $D=2$ and different ensemble sizes N. **b** The same as **a**, but in scaled coordinates

coupling ε increases, the bursts become more and more frequent, and for $\varepsilon=11$ the picture seems to be reversed to that for $\varepsilon=7$: now the mean field is almost always near 1, except for bursts when it is small. The corresponding histograms are presented in Fig. 2.

The averaged value of the mean field $Z=\langle S\rangle$ may be considered as an order parameter for this transition (angle brackets denote averaging over time). Its dependence on the coupling constant is presented in Fig. 3a. Here we report the data for ensembles of different size N. One can see that in ensembles with larger number of elements the transition occurs at larger coupling constants, and the values of Z are smaller. In Fig. 3b the same data are drawn in scaled coordinates, where both Z and ε are scaled by $N^{1/2}$. The curves for different N nearly coincide, so from the numerical data we can conclude that the transition obeys the scaling

$$Z=N^{-1/2}g(\varepsilon N^{-1/2}) \tag{8}$$

with a scaling function g. Below we present simple arguments giving this scaling law (a detailed theory is under study).

As a crude approximation, let us assume that the local potential of the oscillators may be in one of two states – nonexcited (potential U_0) and excited (potential $U_0+\varepsilon SU_1$ with some fixed S). We have already shown that the self-consistent mean-field approach, which is valid for $N\to\infty$, gives the nonexcited state with zero mean field as the only solution. In ensembles with a finite number of elements the mean field deviates from zero due to finite-size fluctuations. According to the law of large numbers, the fluctuations of the mean field are Gaussian with variance proportional to N^{-1}:

$$\mathrm{prob}(S=\bar{S})=\left(\frac{N}{2\pi V}\right)^{1/2}\exp\left(-\frac{N\bar{S}^2}{2V}\right) \tag{9}$$

where $V=2D^{1/2}\Gamma(3/4)/\Gamma(1/4)$ is the variance of the stationary solution (7). Fluctuations with positive $\bar{S}$ produce a bistable potential, and we assume that at this moment the system "switches" to the excited state. In the excited state initially the mean field is positive, and its relaxation back to $S\approx 0$ can be rather slow. The relaxation time can be estimated from the solution of the Fokker-Planck equation for a particle motion in a bistable potential (6a), assuming constant mean field. Changing variables $x=y(\varepsilon S)^{1/2}$, $t=\tau/(\varepsilon S)$ we get from (6a)

$$\frac{\partial W}{\partial \tau}+\frac{\partial}{\partial y}[(y-y^3)W]=R\frac{\partial^2 W}{\partial y^2} \tag{10}$$

with $R=D(\varepsilon S)^{-2}$. The relaxation time for this equation can be estimated as $\tau_r\approx\lambda_1^{-1}(R)$, where λ_1 is the absolute value of the largest non-zero eigenvalue of linear equation (10) [8]. For Eq. (6a), scaling back from τ to t, we get the relaxation (Kramers) time as

$$t_r(\bar{S})\approx[\varepsilon\bar{S}\lambda_1(D\varepsilon^{-2}\bar{S}^{-2})]^{-1}. \tag{11}$$

We can now estimate a characteristic value of the mean field from the condition

$$\mathrm{prob}(s=\bar{S})\cdot t_r(\bar{S})\approx 1$$

meaning that this value is dominated by rare but long-living fluctuations. Combining (9) and (11), and neglecting the prefactor comparing with the exponent in (9), we get for $\bar{S}$ a scaling relation of the form (8).

The critical value of the coupling constant can be also estimated from the cumulant expansion of the nonlinear Fokker-Planck equation (6) [5, 9]. The PDE (6) is equivalent to an infinite set of ODEs for cumulants. The equations for the first two cumulants M_1 and M_2 are [5]:

$$\frac{dM_1}{dt}=\varepsilon SM_1-M_1^3-3M_1M_2-M_3, \tag{12}$$

$$\frac{dM_2}{dt}=D+\varepsilon SM_2-3M_1^2M_2-3M_1M_3-M_4-3M_2^2, \tag{13}$$

where $S=M_1$ (below we will neglect all higher-order cumulants). These equations have the stable fixed point $M_1=0$, $M_2=M_2^0$ corresponding to the stationary solution of (6). For finite ensembles the mean field S fluctuates according to (9). Let us consider the effect of these fluctuations in the linearized equation for the first cumulant (12). Representing the mean field as $S=M_1+(V/N)^{1/2}\eta(t)$, where $\eta(t)$ is Gaussian with zero mean, unit variance and characteristic correlation time t_0, we get a Langevin-type equation

$$\frac{dM_1}{dt}=(\varepsilon(V/N)^{1/2}\eta(t)-3V)M_1. \tag{14}$$

This equation has *multiplicative* noise, so the noise-induced transition can occur for relatively strong noise (small N). Approximating the correlation function of $\eta(t)$ as $\langle\eta(t)\eta(t')\rangle\approx t_0\delta(t-t')$, we get a threshold $\varepsilon_c\approx N^{1/2}(6/t_0)^{1/2}$ in accordance with scaling relation (8).

In conclusion, we have investigated a novel type of transition in the ensemble of globally coupled noisy oscillators. This transition is caused by fluctuations of mean field and in this respect resembles a noise-induced transition. The fluctuations are, however, not given, but should be obtained self-consistently, what makes theoretical study of the problem difficult. The two rather crude approaches presented above give nevertheless correct scaling of the transition.

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Appendix

We describe here a possible experimental realization of the model under consideration. It is an electronic circuit shown in Fig. 4. N identical oscillators are comprised of an inductance L and a 3-polar element with a nonlinear current-voltage characteristics which depends on a governing voltage U: $u_n=g(i_n, U)$. The equations of this

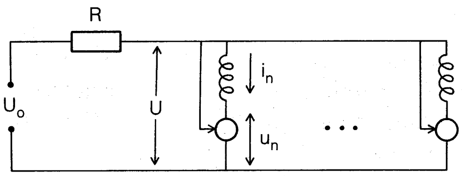

Fig. 4. A sketch of an electronic circuit with a nonlinear coupling through mean field

circuit are

$$L\frac{di_n}{dt}+g\left(i_n,\, U_0-R\sum_1^N i_n\right)+R\sum_1^N i_n=U_0, \qquad n=1, \ldots, N.$$

Taking into account noise acting on each oscillator, we get equations of the type (1). Note that the coupling constant ε is related to the load resistance R as $\varepsilon \sim RN$, so the critical value of R for the transition scales according to (8) as $R_c \sim N^{-1/2}$. If we take 2-polar nonlinear elements, then u_n does not depend on the mean field and it appears in the equations only as a linear term, corresponding to the Komitani-Shimizu model.

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