Do Globally Coupled Maps Really Violate the Law of Large Numbers?

Arkady S. Pikovsky and Jürgen Kurths
Max-Planck-Arbeitsgruppe "Nichtlineare Dynamik," Universität Potsdam, Potsdam, Germany
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Coherent behavior in ensembles of globally coupled maps is investigated in the limit of infinite number of elements. A self-consistent approach based on a nonlinear Frobenius-Perron equation is proposed for such systems, and a possibility of quasiperiodic and chaotic behavior of the mean field is demonstrated. For the study of finite ensembles a noisy nonlinear Frobenius-Perron equation is derived. Previous observations of violations of the law of large numbers are explained.

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Globally coupled nonlinear oscillators are intensively investigated now. Such systems arise naturally in studies of Josephson junction arrays [1], multimode lasers [2], charge-density waves [3], and oscillatory neuronal systems [4]. Also, some unusual properties, such as clustering [5], splay states [6,7], and collective chaotic behavior [5,8-10], were predicted for these systems theoretically.

Several basic models of coupled oscillators were proposed. The behavior of coupled Josephson junctions and limit-cycle oscillators may be described by coupled ordinary differential equations [5,6]. For noisy bistable oscillations a description based on the Fokker-Planck equation was proposed in [11]. In the case when individual oscillators are chaotic, a model of globally coupled maps can be used [12,13]. In particular, Fabiny and Wiesenfeld showed [14] that such a model describes adequately coupled electrical circuits comprised of p-n junctions.

Recent studies of globally coupled maps have revealed some rather surprising results; e.g., in Refs. [15,16] it was shown that globally coupled maps violate in some cases the law of large numbers, due probably to nontrivial coherent behavior. In this paper we use a statistical approach analogous to that applied previously to coupled noisy oscillators [11] to study the collective behavior in large ensembles of globally coupled maps. In the limit when the number of the elements N tends to infinity, we describe the system self-consistently with a nonlinear Frobenius-Perron equation, which is considered then as a nonlinear dynamical system with an infinite number of degrees of freedom. The behavior of the system depends on a coupling constant and exhibits a transition to chaos via quasiperiodicity. For finite N the mean field obeys additional random fluctuations; in this case the system can be described by a noisy nonlinear Frobenius-Perron equation.

Let us consider an ensemble of N identical discrete-time nonlinear oscillators,

$$x_{t+1}^{(i)}=f(x_t^{(i)},a),\quad i=1,\ldots,N\,, \tag{1}$$

depending on a parameter a. We suppose that these oscillators are coupled through the mean field s defined as

$$s_t=\frac{1}{N}\sum_{i=1}^{N}x_t^{(i)}\,. \tag{2}$$

We assume that the coupling comes in (1) through a dependence of the parameter a on s_t:

$$a_t=a^0+\varepsilon s_t\,, \tag{3}$$

where ε is the coupling constant and a^0 is a parameter value of uncoupled maps.

The ensemble governed by Eq. (1) can be characterized by its probability distribution density $W_t(x)$, whose evolution obeys the Frobenius-Perron equation [17]

$$W_{t+1}(x)=\int dy\,\delta(x-f(y,a))W_t(y)\,. \tag{4}$$

If we take into account that according to (3) the Frobenius-Perron operator depends on W_t, we obtain the nonlinear equation

$$W_{t+1}(x)=\int dy\,\delta(x-f(y,a_t))W_t(y)\,, \tag{5}$$

$$a_t=a^0+\varepsilon s_t\,, \tag{6}$$

$$s_t=\int xW_t(x)dx\,, \tag{7}$$

which we call the nonlinear Frobenius-Perron equation (NFPE). It is completely analogous to the nonlinear Fokker-Planck equation derived in [11] for ensembles of coupled noisy continuous-time oscillators. [Note that for generalization of the system (5)-(7) to the case of noisy discrete oscillators one has only to modify the kernel in the Frobenius-Perron operator (5).]

Let us start with looking for the stationary solutions of Eqs. (5)-(7) meaning that $s_t=s$ does not depend on t. Then, if the map $f(x,a)$ is mixing, Eq. (4) with constant a describes an evolution to a unique stationary distribution that depends on the parameter a:

$$\lim_{t\to\infty}W_t(x)=W_\infty(x,a)\,. \tag{8}$$

Hence, the mean field also depends on a,

$$s=\int xW_\infty(x,a)dx\equiv F(a)\,, \tag{9}$$

and from Eq. (6) we get an equation for a,

$$a=a^0+\varepsilon F(a)\,. \tag{10}$$

If $F(a)$ is a bounded continuous function, Eq. (10) has at

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least one solution that tends to a^0 for $\varepsilon \to 0$ (uncoupled oscillators). However, in many widely discussed systems like logistic map $f(x,a) = a - x^2$, the function $F(a)$ is discontinuous (due to the fact that periodic windows are everywhere dense on the parameter interval) and a stationary solution may not exist for some values of a^0 and ε [18]. In the case of the logistic map there is also another difficulty: Because of the existence of periodic windows and bands the map is not mixing. If the parameter a is chosen within a periodic window, Eq. (8) is no more valid and the Frobenius-Perron equation (4) has a continuous set of periodic invariant solutions. Indeed, if $x_1, x_2, \ldots, x_m$ is a stable cycle, then $W(x) = \sum_{i=1}^m \kappa_i \times \delta(x - x_i)$ is a periodic solution of Eq. (4) for any set $\kappa_1, \ldots, \kappa_m$ satisfying the normalization $\kappa_1 + \cdots + \kappa_m = 1$. These periodic solutions may be responsible for a nonstatistical behavior numerically observed in ensembles of coupled logistic maps in [15,16,18]. If, however, a small noise is added to the logistic map, Eq. (4) has a unique invariant solution like for the mixing mappings. Then a stationary solution of Eq. (10) exists at least for small enough ε. For large ε this solution may become unstable and a more complicated behavior is observed.

As a concrete example for nontrivial collective behavior let us consider a tent map

$$f(x,a) = a(1 - |x|) - 1 . \tag{11}$$

This map is mixing for $\sqrt{2} < a < 2$ and in the calculations presented below a_t is always in this interval. Direct numerical simulation of the system (5)-(7) was performed with a finite-difference scheme with 4000 nodes in the interval $[-1,1]$; the evolution of a randomly chosen initial density was followed. After transients were over, we analyzed the sequence $\{s_t\}$. Typically, with an increase of the coupling constant ε (which may be either positive or negative) a complex sequence of bifurcations is observed. Referring for a detailed bifurcation diagram to [19], we present here the two most interesting examples. For $a_0 = 1.9$, $\varepsilon = -0.74$ the attractor generated appears to be quasiperiodic [see Fig. 1(a); the largest Lyapunov exponent is nearly zero in this case]. For $a_0 = 1.9$, $\varepsilon = -1$ a chaotic attractor is observed with at least one positive Lyapunov exponent [see Fig. 1(b)].

The NFPE describes the evolution of the ensemble in the limit $N \to \infty$. For finite N we still can use Eqs. (5) and (6), but with Eq. (2) instead of (7). If we assume that the law of large numbers is valid for our system, then we can rewrite Eq. (2) as

$$s_t = \bar{x}_t + \frac{D_t^{1/2}}{N^{1/2}} \xi_t , \tag{12}$$

where $\bar{x}_t = \int x W_t(x) dx$; $D_t = \int (x - \bar{x}_t)^2 W_t(x) dx$ and ξ_t are Gaussian random variables with zero mean and unit variance. The system (5), (6), and (12) is in fact a *noise-driven* nonlinear evolution equation. One can expect that for a small noise amplitude (large ensemble size N) the regimes observed in a noise-free system (5)-(7) are only slightly modified. The results of direct numerical simulation of system (1) and (2) confirm this; see Fig. 2.

Note that here we use the law of large numbers in a more precise sense than the authors of [15,16]. Indeed, the law of large numbers can be applied to an ensemble of *equally* distributed random variables. Thus, if the distribution function depends explicitly on time, averages over time and ensemble are not equivalent [20]. While the variance V of the fluctuations of the mean field averaged over the ensemble is proportional to N^{-1}, the time-averaged variance saturates for large N at the value $V_\infty = \langle (s_t - \langle s_t \rangle)^2 \rangle$, where s_t obeys the noiseless system (5)-(7). Only if the NFPE has a stable fixed point solution, one has $V_\infty = 0$ and both methods of averaging are equivalent. This explains why, using the time averaging in Refs. [15,16], violations of the law of large numbers were observed.

The proper procedure to confirm the law of large numbers for the fluctuations of the mean field (2) is the following. The observed time sequence s_t must be considered as an output of a noisy dynamical process, governed by a system (5), (6), and (12). Therefore one has to apply to this time series one of the methods of noise estimation in dynamical systems [21,22]. These methods include embedding in a phase space, and for complex regimes such as shown in Figs. 1 and 2 are rather computer time consuming. So we applied a simple

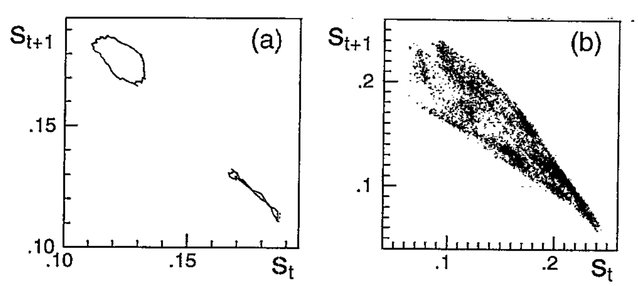

FIG. 1. Two-dimensional plot of successive iterations of the NFPE (5)-(7), (11). (a) $\varepsilon = -0.74$, quasiperiodic regime; (b) $\varepsilon = -1$, chaotic regime.

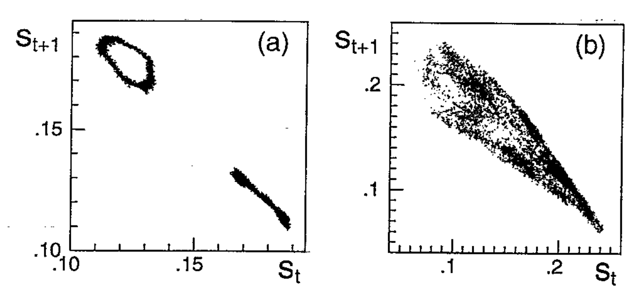

FIG. 2. Direct simulations of an ensemble of $N = 10^6$ oscillators (1) and (2) for the same parameters as in Fig. 1.

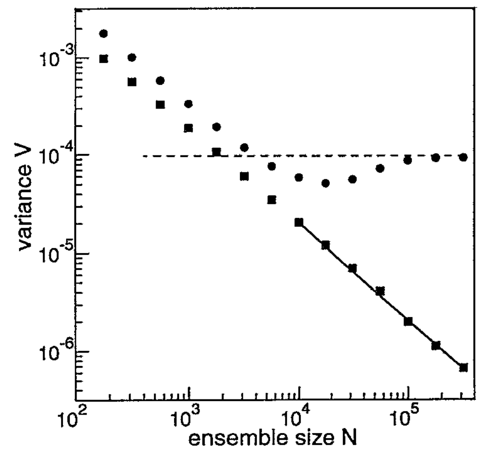

FIG. 3. Estimations of the effective noise for the fluctuations of the mean field in the models (1) and (2) for coupled tent maps with $\varepsilon = -0.44$. Circles, time averaging; squares, dynamical noise estimation. Dashed line, variance V_∞ of period-two cycle obtained from the NFPE (5)–(7). Solid line has slope -1.

method [23] of noise intensity estimation to the case $\varepsilon = -0.44$ when a period-two cycle in the NFPE is observed. [Having a time series from (1) and (2), we took all points s_τ falling in a small interval (typically of width 10^{-4}) and estimated noise intensity from the spreading of their images $s_{\tau+1}$. After averaging over different intervals we got the value for the noise variance.] Figure 3 shows that while methods of Refs. [15,16] saturate for $N \to \infty$ at the predicted value V_∞, the dynamical noise estimation method gives N^{-1} dependence of noise variance, in accordance with the law of large numbers. Note that recently reported violation of the law of large numbers in an ensemble of uncoupled oscillators governed by the same noise [24] was explained in Refs. [20,25,26] with essentially the same arguments as above.

In conclusion, using the self-consistent approach based on the nonlinear Frobenius-Perron equation we have shown that ensembles of globally coupled nonlinear discrete-time oscillators may exhibit quasiperiodic and chaotic coherent collective behavior. We have considered only mean-field coupling, but our approach can be easily generalized to any global coupling. We have also shown that the law of large numbers is valid for systems with mixing chaotic attractors, if implemented properly. It seems suggestive to apply this approach to *local* coupling in *high-dimensional* lattices of nonlinear oscillators and cellular automata, where quasiperiodic collective behavior has been observed recently [27–29].

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