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Collective behavior in ensembles of globally coupled maps

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Abstract

Coherent collective behavior in an ensemble of globally coupled maps is investigated in the limit of infinite number of elements. A nonlinear Frobenius–Perron equation is derived for this system, and it is shown that it can have quasiperiodic and chaotic solutions. For the description of finite ensembles we propose a noisy nonlinear Frobenius–Perron equation and show that it gives the correct power spectrum of mean field fluctuations.

1. Introduction

In recent years much work has been devoted to the studies of nonlinear systems with many degrees of freedom. One of the most popular models in this field is an ensemble of globally coupled nonlinear oscillators. Such systems arise naturally in the description of Josephson junction arrays [1], multimode lasers [2], charge-density waves [3], and oscillatory neuronal systems [4]. More generally, global coupling appears as a result of mean-field approach to the dynamics of distributed systems. One can consider the interaction of deterministic or random (chaotic) oscillators. In the former case studies of ensembles of nonlinear continuous-time oscillators revealed such interesting features as clustering [5], existing of splay states [6,7], and even chaotic collective behavior [5,8–10]. For ensembles of irregular individual systems, different models were proposed. The interaction of chaotic oscillators is usually modeled with ensembles of globally coupled map [11–13], Fabiny and Wiesenfeld showed [14] that such a model describes adequately coupled electrical circuits comprised of p–n junctions. If individual elements are not intrinsically chaotic, but are randomized by external noise, systems of noise-driven oscillators are used [15–18]. For the case of irregular individual systems, main efforts were devoted to studies of nontrivial collective behavior.

For the description of collective behavior in ensembles of continuous-time noise-driven oscillators, a self-consistent mean-field approach was proposed by Desai and Zwanzi [16]. The collective dynamics is described by a *nonlinear* Fokker–Planck equation, which is exact in the thermodynamic limit $N \to \infty$ (N is the number of oscillators) [16,19]. The solutions of this equation can obey bifurcations that correspond to phase transitions in the syste [16,18].

Ensembles of globally coupled chaotic maps were studied in Refs. [20,21,13], where "violation of the law of large numbers" in these systems was described: in some cases the calculated variance of the fluctuations of the mean field did not scale as N^{-1} for large N. In this paper, we apply the self-consistent mean-field approach similar to that of Desai and Zwanzig

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to these systems (recently an analogous analysis has been performed in Ref. [22] for coupled homographic maps). In the thermodynamic limit, we describe in Section 2 the evolution of the probability distribution density of the ensemble with a nonlinear Frobenius–Perron equation. In Section 3 we study in detail the case of coupled tent maps. We describe bifurcations in the Frobenius–Perron equation and show that collective periodic, quasiperiodic and chaotic behavior can be observed in the system. We also discuss in Section 4 difficulties in the application of this method to coupled logistic maps. In Section 5 we propose to describe ensembles with finite number of elements by means of a *noisy* nonlinear Frobenius–Perron equation and compare its solutions with results of direct modeling of such ensembles. This allows to explain the observed "violation of the law of large numbers". We also show how to proceed the data analysis without contradictions with the law of large numbers. We summarize the results in Section 6.

2. Nonlinear Frobenius–Perron equation

The nonlinear model, we investigate here, is a system of N identical discrete-time oscillators. An oscillator is described by the variable $x(i)$ which obeys the equation

$$x_{t+1}^{(i)} = f(x_t^{(i)}, a), \quad i = 1, ..., N, \tag{1}$$

depending on the parameter a. We suppose that these oscillators are coupled through the mean field s defined as

$$s_t = \frac{1}{N} \sum_{i=1}^{N} x_t^{(i)}. \tag{2}$$

We assume that the coupling comes in (1) through a dependence of the parameter a on s_t:

$$a_t = a^0 + \epsilon s_t\,, \tag{3}$$

where ϵ is the coupling constant and a^0 is the parameter value of the uncoupled maps.

The ensemble governed by Eq. (1) can be characterized by its probability distribution density $W_t(x)$, whose evolution obeys the Frobenius–Perron equation [23]

$$W_{t+1}(x) = \int dy\, \delta(x - f(y, a))\, W_t(y)\,. \tag{4}$$

If we take into account that according to (3) the Frobenius–Perron operator depends on the mean field, we obtain the nonlinear equation

$$W_{t+1}(x) = \int dy\, \delta(x - f(y, a_t))\, W_t(y)\,, \tag{5}$$

$$a_t = a^0 + \epsilon s_t\,, \tag{6}$$

$$s_t = \int x\, W_t(x)\, dx\,, \tag{7}$$

which we call a nonlinear Frobenius–Perron equation (NFPE). It is completely analogous to the nonlinear Fokker–Planck equation derived in [16] for ensembles of coupled noisy continuous-time oscillators. Note that for generalization of the system (5)–(7) to the case of noisy discrete oscillators one has only to modify the kernel in the Frobenius–Perron operator (5).

Let us emphasize that so far we did not use any assumptions about the properties of the mapping f. If the mapping (1) demonstrates regular (periodic) behavior, then singular solutions (δ–functions), that follow stable periodic orbits, appear in Eq. (4) (see Section 4). In the rest of this section we shall assume that the mapping f is chaotic and, moreover, mixing. In this case, the stationary solution of NFPE is unique and can be easily found numerically. For a mixing map $f(x,a)$, Eq. (4) with constant a describes an evolution to a unique stationary distribution that depends on the parameter a:

$$\lim_{t\to\infty} W_t(x) = W_\infty(x, a). \tag{8}$$

From this distribution the mean field may be calculated as

$$s = \int x\, W_\infty(x, a)\, dx \equiv F(a) \tag{9}$$

and from Eq. (6) an equation for a is obtained:

$$a = a^0 + \epsilon F(a). \tag{10}$$

If $F(a)$ is a bounded continuous function, Eq. (10) has at least one solution that tends to a^0 for $\epsilon \to$

0 (uncoupled oscillators). It may also happen that Eq. (10) has several solutions. The function $F(a)$ is typically non-smooth [24], but it is not clear if it can be discontinuous for maps that are mixing in the whole range of parameter a variations. In the latter case for some ϵ there may be no stationary solution of NFPE at all.

As a first approximation to the time evolution in the full system (5)–(7), we can use Eq. (10). If we assume that already after one application of the Frobenius–Perron operator a probability density is equal to its stationary value (8), we can approximate Eq. (5) as

$$W_{t+1} \approx W_{\infty}(x, a_t)$$

to obtain

$$a_{t+1} = a^0 + \epsilon F(a_t). \qquad (11)$$

We call this approximation quasi-static, it has been recently independently suggested in Ref. [22]. A similar equation was derived in Ref. [25] for a nonlinear network of synchronous threshold elements. The mapping (11) may have periodic or even chaotic solutions, which correspond to a nontrivial collective behavior of the ensemble of coupled maps. However, generally one cannot expect that the quasi-static approximation, which is essentially the one-dimensional approximation of the infinite-dimensional system (5)–(7), is satisfactory.

3. Coupled tent maps

Let us consider an ensemble of coupled tent maps

$$f(x, a) = a(|x| - 1) - 1. \qquad (12)$$

This map is mixing for $-\sqrt{2} > a > -2$ and in the calculations presented below, a_t is always in this interval. For the tent map the dependence of the mean field on the parameter a is presented in Fig. 1. The function $F(a)$ appears to be continuous, albeit not smooth; it looks like a logistic function. Thus, one can expect to find in the quasi-static approximation (11) a transition to chaos through period doublings. We, however,

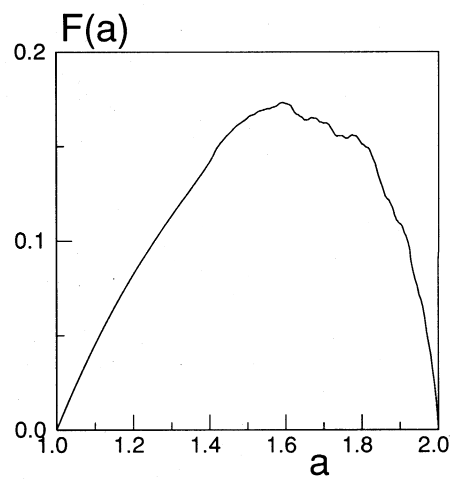

Fig. 1. Averaged field vs. parameter a for the tent map (12).

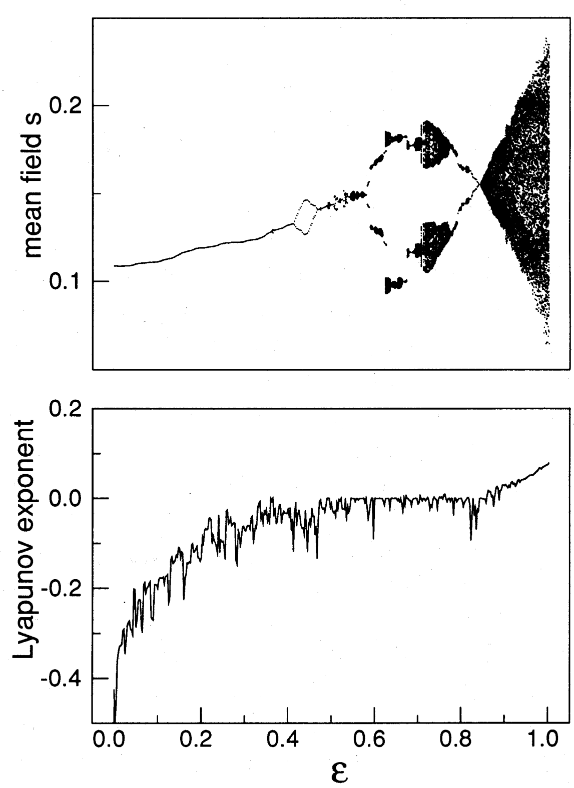

Fig. 2. Bifurcation diagram of the NFPE for the tent map. (a) mean field vs. parameter ϵ; (b) Lyapunov exponent vs. parameter ϵ.

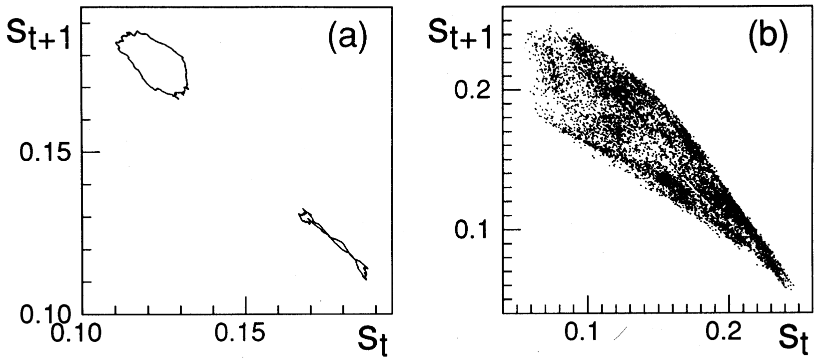

Fig. 3. Two-dimensional plot of successive iterations of the NFPE (5)–(7), (12). (a) $\epsilon = 0.74$, quasiperiodic regime; (b) $\epsilon = 1$, chaotic regime.

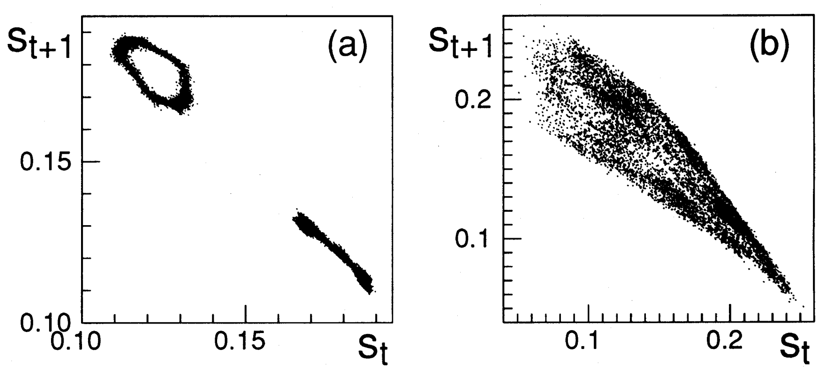

Fig. 4. Direct simulations of an ensemble of $N = 10^6$ oscillators for the same parameters as in Fig. 2.

have not iterated (11) with $F(a)$ from Fig. 1, but have solved the full NFPE for this system numerically, using a finite-difference scheme with several thousands nodes in the interval $[-1, 1]$. Below we present the results of calculations for a particular value of the parameter $a^0 = -1.9$.

For sufficiently small coupling constants, a stable stationary state described by Eq. (10) is established. This solution loses its stability at $\epsilon \approx 0.415$ with creation of period-2 oscillations. With further increasing of the coupling constant, a complex sequence of transitions between periodic, quasiperiodic and chaotic states is observed (see Fig. 2). In Fig. 3, phase portraits of quasiperiodic and chaotic regimes are presented. The presence of quasiperiodic regimes means, in particular, that the quasi-static approximation fails for this system, because quasiperiodic motions are impossible on one-dimensional mappings of the type (11).

We see that in the case of coupled tent maps the nonlinear Frobenius–Perron equation demonstrates properties, typical for dissipative nonlinear systems with many degrees of freedom. The nontrivial collective behavior of the ensembles corresponds to solutions which are more complex than a fixed point.

We have also performed direct modeling of very large ensembles of tent maps and report the results in Fig. 4. One can see that the NFPE correctly describes the dynamics. The discrepancy, which is clearly seen for the quasiperiodic regime, is caused by the remaining for finite N statistical fluctuations; we shall discuss them in Section 5 below.

4. Coupled logistic maps

In this section we study the logistic map of the form

$$f(x,a) = a - x^2. \tag{13}$$

As the parameter a increases, this map demonstrates transition to chaos through period-doublings, and a very complicated structure of periodic windows for larger values of a. It was proven that the set of points a, for which this map is chaotic, has a positive measur [26], but we cannot expect that for an arbitrary chosen point the map will be mixing. Moreover, if the parameter a^0 is chosen within a periodic window, the Frobenius–Perron equation has already for $\epsilon = 0$ an infinite number of solutions (see also discussion in [27]). Indeed, if Eq. (13) has a stable period-m cycle $\{x_1, x_2, ..., x_m\}$, then

$$W(x) = \kappa_1\delta(x - x_1) + \kappa_2\delta(x - x_2) + \cdots + \kappa_m\delta(x - x_m) \tag{14}$$

is an m-periodic solution of the Frobenius–Perron equation for any set of coefficients $\{\kappa_k\}$ satisfying the normalization condition $\kappa_1 + \cdots + \kappa_m = 1$. For such solutions the mean field s_t also oscillates with period m. Because periodic windows have finite width in a, the same periodic behavior will be observed for sufficiently small coupling constants. Suppose now that we choose the value of a^0 for which the map (13) is chaotic. In any vicinity of this parameter value there are periodic windows. Let us denote the center of one of these windows by a^0_*, and the mean value obtained from (9),(14) with $\kappa_1 = \kappa_2 = ... = \kappa_m$ by $F(a^0_*)$. Then, as follows from (10), the NFPE has a stationary solution (14) if $\epsilon = (a^0_* - a^0)F^{-1}(a^0_*)$. A small perturbation of the values $\{\kappa_k\}$ will immediately produce a periodic solution. We see that coupling of chaotic logistic maps may lead to a non-chaotic behavior of individual maps and to periodic collective oscillations.

The arguments presented above show that the dynamics of globally coupled logistic maps is very complicated, due to high sensitivity of the dynamics of individual map to variations of the parameter. In this case the NFPE, considered as a dynamical system, has very unusual properties. These features also make the numerical treatment of the NFPE extremely difficult, and it is not surprising that numerical simulations in Ref. [13] gave no reliable results. It seems, that the only way to regularize the model is to include external noise in the mapping (13). This is, however, beyond the scope of this paper.

5. Finite ensembles

The nonlinear Frobenius–Perron equations (5)–(7) describe globally coupled maps in the limit $N \to \infty$. For finite N the representation of the mean field through the integral of the distribution function (7) is no more valid. We can, however, use the central limit theorem, which states that if N independent random variables $x(i)$ have the same distribution $W(x)$, then for large N the mean field $N^{-1}\sum x(i)$ has approximately the Gaussian distribution with mean $\bar{x} = \int dx\, xW(x)$ and variance $N^{-1}\int dx\,(x-\bar{x})^2W(x)$. So, we can write

$$s_t = \frac{1}{N}\sum_{i=1}^{N} x_t^{(i)} = \bar{x}_t + V^{1/2}\eta_t, \tag{15}$$

where

$$\bar{x}_t = \int dx\, xW_t(x),$$

$$V = N^{-1}\int dx\,(x-\bar{x}_t)^2W_t(x)$$

and η_t is a gaussian random number with zero mean and unit variance. Eq. (15) together with (5),(6) can be used to describe of the evolution of finite ensembles of globally coupled maps. It is a *noisy nonlinear Frobenius–Perron equation.* In order to define it completely, we have to know the correlation function of the process η_t. We shall discuss this for the simplest case of a small coupling constant for the tent map, when the nonlinear Frobenius–Perron equation has a stable fixed point solution.

5.1. Spectrum of mean field fluctuations

Let us consider a finite ensemble of coupled tent maps (12) with $a^0 = -1.9$ and ϵ small enough, so that the NFPE has a stable fixed point $a = a_c$. The lowest-order approximation in applying Eq. (15) is to neglect all correlations in η_t. Then, we can easily simulate the noisy nonlinear Frobenius–Perron equations (5),(6),(15) and compare the power spectrum of the in this way obtained mean field $S_1(\omega)$ with the spectrum $S(\omega)$ which we get from direct numerical simulations of the system (1)–(3). While the variance of mean field fluctuations is quite close to the observed value, both power spectra $S_1(\omega)$ and $S(\omega)$ differ significantly (Fig. 5). This means that we cannot neglect correlations in the fluctuations of the mean value η_t. The simplest way to account these correlations is to calculate them in the thermodynamic limit $N \to \infty$. In this limit, all oscillators are governed by the same tent map with parameter value a_c. Therefore, the average s has the same correlation function as an individual oscillator (unfortunately, for the tent map with $a \neq 2$ we can obtain this correlation function only numerically). The process η_t then should be constructed with this correlation function, we denote its spectrum as $S_x(\omega)$. We can make at this point one more approximation. Because for large N the amplitude of noise in the noisy nonlinear Frobenius–Perron equation is small, we can consider the second term in the r.h.s. of Eq. (15) as a small perturbation. So the fluctuations of the mean field can be considered as a result of linear transformation of the "input" noise η_t:

$$S_{mf}(\omega) = K(\omega) S_\eta(\omega).$$

Here $S_\eta(\omega)$ is spectrum of η_t, and $K(\omega)$ is a "transfer function" of NFPE. This transfer function can be obtained from the calculations of noisy NFPE with uncorrelated noise η_t: in this case $S_\eta(\omega) = \text{const.}$, so $K(\omega) = S_1(\omega)$. In the improved approximation, when we use for $S_\eta(\omega)$ the spectrum $S_x(\omega)$, we therefore get

$$S_2(\omega) = K(\omega) S_x(\omega) = S_1(\omega) S_x(\omega).$$

The spectrum calculated in this way is in a rather good accordance with $S(\omega)$, obtained from direct numerical simulations of a large ensemble (see Fig. 5).

Although the crude method described above gives rather satisfactory results, we at present can neither justify it (considering, e.g., higher approximations) nor extend (to the cases when the NFPE has solutions more complicated then a fixed point). Probably,

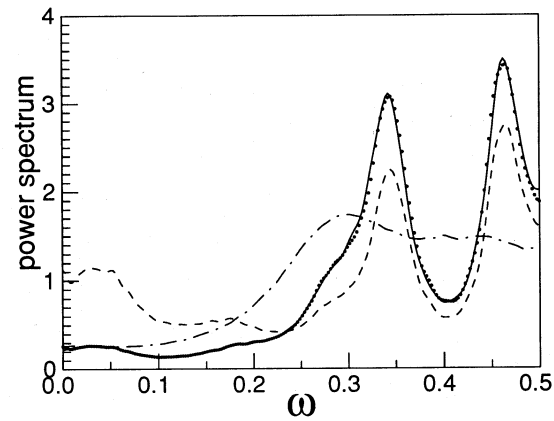

Fig. 5. Power spectrum of the mean field fluctuations. Full line: $S(\omega)$-results of direct calculations of the ensemble of $N = 10000$ tent maps for $a^0 = -1.9$, $\epsilon = 0.3$; dashed line: $S_1(\omega)$-output of noisy nonlinear Frobenius–Perron equation with uncorrelated η_t; dash-dotted line: $S_x(\omega)$-spectrum of an individual tent map; dots: $S_2(\omega) = S_1(\omega) S_x(\omega)$.

considering more simple models (e.g., coupled homographic maps, for which the NFPE can be solved analytical [22]) will provide better comprehension of the correlation properties of mean field oscillations.

5.2. The law of large numbers

Let us now discuss violations of the law of large numbers reported for globally coupled maps in Refs. [20,13]. Mainly this phenomenon was studied for ensembles of logistic maps. However, as we discussed in Section 4, it is very difficult to determine the behavior of these maps in the thermodynamic limit (if this limit exists at all). So, consider the violation of the law of large numbers for coupled tent maps. In Refs. [20,13] the following procedure was used for calculations of the variance of mean field fluctuations: having a time series of the mean field s_t, one calculates its average $\overline{s}$ and variance $\overline{\hat{s}^2}$ using simple time averaging:

$$\overline{s} = \frac{1}{T}\sum_{1}^{T} s_t, \qquad \overline{\hat{s}^2} = \frac{1}{T}\sum_{1}^{T}(s_t - \overline{s})^2. \qquad (16)$$

The violation of the law of large numbers is inferred from the fact, that $\overline{\hat{s}^2}$ does not decrease as N^{-1}, but saturates for large N at some value V_∞. From our ap-

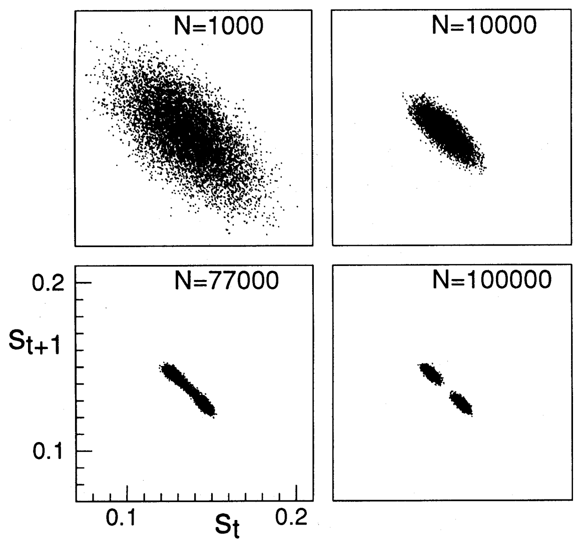

Fig. 6. Dynamics of mean field in ensembles of tent maps with $a^0 = -1.9$, $\epsilon = 0.44$, when the NFPE has a period-2 solution.

proach, this is explained as a manifestation of nontrivial collective behavior in the thermodynamic limit. In this limit the sequence s_t represents a nonlinear dynamical behavior of the system (5)–(7), and using the procedure (16) one estimates the "width" of the attractor. For example, if the attractor is a period-two orbit (see Fig. 6) ..., s_1, s_2, s_1, s_2, ..., then the method (16) gives $\overline{s^2} = (s_1 - s_2)^2/4$.

If the attractor is not a fixed point, but a periodic orbit, a quasiperiodic, or a chaotic set, this "width" is non-zero, and appears as a saturation value V_∞. Thus, the procedure (16) is inadequate in the cases of nontrivial collective behavior. Instead, we suggest to check the validity of relation (15), or, in other words, to check if the system can be considered as governed by a noisy nonlinear Frobenius–Perron equation with a noise intensity scaling as N^{-1}. So, we should consider the time series s_t as an output of a noisy dynamical system, and try to estimate the noise level. This is one of the common tasks in the nonlinear time series analysi [28,29]. Mainly, one is interested in the noise reduction of chaotic (or, more generally, dynamical) time series data. However, application of a cleaning procedure gives the level of noise as a by-product. For a general review on the noise-reduction we refer to a recent paper [29], where many common methods are described. Moreover, the section 6 of this paper describes "How much noise is taken out", what is exactly our task. We outline below the method we actually used (see also [33]).

Given the time series $\{s_t\}$, we proceeded in the following steps. First, using the Takens time-delay-embedding method, we constructed a series of vectors $\sigma_t = (s_t, s_{t-1}, ..., s_{t-m+1})$, where m is the embedding dimension. Then, for a given vector σ_t we calculated a "predictor" (noiseless vector) $\tilde{\sigma}_{t+1}$. This is done by taking all vectors lying in a small neighbourhood of σ_t, and averaging their images. The difference between the cleaned value $\tilde{\sigma}_{t+1}$ and the observed one σ_{t+1} give the noise amplitude at time t. Averaging these quantities, we obtained finally the estimate of the noise level in the system. In principle, one should consider sufficiently large embedding dimensions, but in high dimensions the number of neighbours of each point is small. Thus, practically this method works well only for simple (low-dimensional) attractors. Therefore, we demonstrate it not for rather complicated regimes, as shown in Figs. 3,4, but for the simplest period-2 case. For the noisy period-2 oscillations (Fig. 6) the one-dimensional embedding is sufficient, and we are able to get agreement with the law of large numbers, see Fig. 7. For the chosen parameters values, in the thermodynamic limit there are period-two oscillations $s_1 = 0.1268$, $s_2 = 0.1465$. Simple averaging gives saturation exactly at the predicted value of variance, while our method of noise estimation gives fluctuations values decaying as N^{-1}.

6. Concluding remarks

In conclusion, we have shown that ensembles of globally coupled nonlinear discrete-time oscillators may exhibit quasiperiodic and chaotic coherent collective behavior. We have also argued that the law of large numbers is valid for systems with mixing chaotic attractors, if implemented properly.

We have used the self-consistent mean-field approach, derived previously for systems of globally coupled noisy continuous-time oscillators. For these systems in the thermodynamic limit $N \to \infty$ one ob-

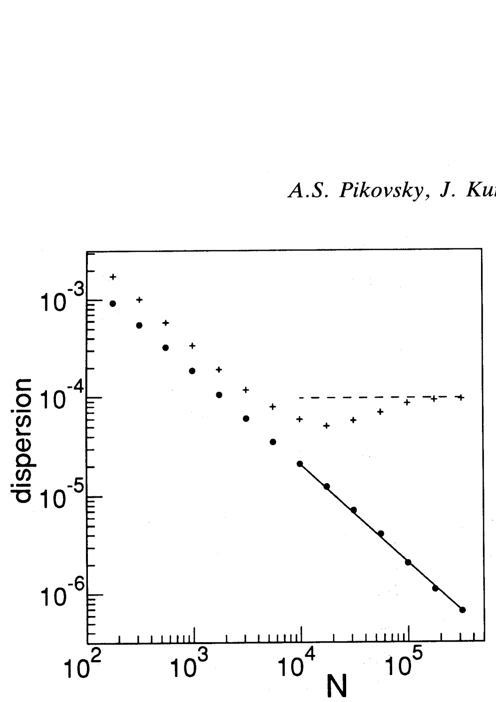

Fig. 7. Variance of mean field fluctuations vs. ensemble size N (for the same parameters as in Fig. 6). Pluses: time averaging (16), circles: estimation of noise following the method described in the text. Dashed line: saturation value of $\overline{s^2}$ obtained from the solution of NFPE.

tains a nonlinear partial differential equation (nonlinear Fokker–Planck equation), while we for discrete-time oscillators have derived a nonlinear functional mapping (nonlinear Frobenius–Perron equation). In both equations nontrivial bifurcations can be obtained, but in the later case, due to discreteness of time, transition to chaotic collective behavior is easier to be observed.

Near the thermodynamic limit, for large but finite N, we proposed to describe the system with a noisy Frobenius–Perron equation. This approach is not fully self-consistent, because correlation properties of the mean field fluctuations are not known in advance. At present we have only an approximate iterative method of calculation of the correlations, that works satisfactory at least in simple cases.

Our main example was the system of coupled tent maps. These maps are mixing and therefore the nonlinear Frobenius–Perron equation is well defined, contrary to the case of logistic maps. One disadvantage is that the Frobenius–Perron equation can be solved only numerically. This disadvantage has been recently overcome by Griniasty and Haki [22], who applied the same approach to an ensemble of homographic maps, for which the Frobenius–Perron equation can be solved analytically.

The analysis of application of the law of large numbers to the globally coupled oscillators, performed above, shows that for such systems time and ensemble averaging do not coincide. This can be inferred already from the analysis of a simpler situation – an ensemble of uncoupled oscillators governed by the same noise. The violation of the law of large numbers reported for this system [30] was explained in Refs. [22,31,32] with essentially the same arguments as we used in Section 5 above.

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