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Correlations and spectra of strange non-chaotic attractors

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Abstract. We consider correlations and spectra of strange non-chaotic attractors in quasiperiodically driven nonlinear systems. It is demonstrated that a self-similar autocorrelation function and a singular continuous spectrum may be observed in such systems. The properties of correlations and spectra depend very subtly on the rotation number of the quasiperiodic force and on the parameters of the system, for some parameter values the usual discrete spectrum is restored.

1. Introduction

Strange non-chaotic attractors typically appear in quasiperiodically forced nonlinear systems. These attractors were first described by Grebogi *et al* in 1984 [1] and since then investigated in a number of numerical [2–7] and experimental [8, 9] studies. A typical system considered in most of these works is a nonlinear oscillator with quasiperiodic (two-frequency) forcing. With increase of the force amplitude, a transition to chaos, when the largest Lyapunov exponent becomes positive, is usually observed, and strange non-chaotic attractors appear in a region of parameters just below this transition. SNAs exhibit some properties of regular as well as chaotic systems. Like regular attractors they have only negative and zero (connected to quasiperiodic forcing) Lyapunov exponents, like chaotic strange attractors they are fractals. Both these properties are rather difficult to verify in numerical, and moreover, in real experiments. One very common tool in investigation of complex regimes is calculation of the autocorrelation function and the power spectrum. In the case of periodic (quasiperiodic) oscillations the spectrum is discrete and the autocorrelation function returns (almost exactly) to 1. For chaotic oscillations the spectrum has a continuous component, and the autocorrelation function decreases.

In this paper we focus just on the correlation and spectral properties of SNAs. Our main result is that SNA can have singular continuous spectrum. This spectrum is intermediate between regular and random, and has been recently investigated in some models of quasiperiodic lattices and quasiperiodically forced quantum systems [10–13]. We will present here only the phenomenology of the autocorrelation function and the spectrum, based on numerical analysis; a renormalization group study is now in progress [14].

The paper is organized as follows. In section 2 we describe the basic models of SNA we will deal with. In sections 3 and 4 the properties of autocorrelation function and spectrum are discussed. In section 5 we compare our findings with other studies of spectra of SNAs, as well as with other cases of singular continuous spectra. In appendix A the necessary properties of quasiperiodic functions are described, while in appendix B the properties of correlations and spectra of usual quasiperiodic motion are summarized.

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2. Basic models

Our basic model will be the SNA first proposed by Grebogi *et al* [1] and then studied in [7]. The system is described by the map

$$x_{n+1} = f(x_n, \theta_n) = 2\sigma(\tanh x_n)\cos(2\pi\theta_n) \quad (1)$$

$$\theta_{n+1} = \theta_n + \omega \quad \mathrm{mod}\ 1 . \quad (2)$$

One can consider (1) as a quasiperiodically forced map, and the external force is multiplicative. The frequency of the forcing is given by the parameter ω (rotation number of the map (2)), which should be irrational. The parameter σ should be larger then 1 for the SNA to exist [1], otherwise the value of σ appears not to influence the qualitative properties of the system, so we will fix it at $\sigma = 1.5$ as in [1]. We will refer to the system (1) and (2) as 'model A'. In the previous papers [1, 7] only 'the most famous' rotation number—the golden mean $\omega = \omega_{gm} = (\sqrt{5} - 1)/2$—was used. In this paper we will also study other irrationals. The properties of irrationals and their resonances are described briefly in appendix A.

We will also consider two generalizations of the map (1). The first, where the quasiperiodic force acts both multiplicatively and additively

$$x_{n+1} = f(x_n, \theta_n) = 2\sigma(\tanh x_n)\cos(2\pi\theta_n) + \alpha\cos(2\pi(\theta_n + \beta)) \quad (3)$$

has already been investigated in [1, 7]. In [7] it was shown that the SNA exists only for $\alpha = 0$, although for small α it is rather difficult to distinguish the existing smooth torus from a SNA. We will use this example (referred to as model B) to discuss differences in the spectra of usual quasiperiodic motions and SNA.

Another generalization is

$$x_{n+1} = f(x_n, \theta_n) = 2\sigma(\tanh x_n)(\cos(2\pi\theta_n) + \gamma) . \quad (4)$$

Here the quasiperiodic force remains purely multiplicative, but now has a zero-frequency component, proportional to γ. Using the same arguments as in [1], one can show that the SNA in the system (4) and (2) (referred to as model C) exists also for non-zero γ. We will see, however, that the properties of correlations and spectra depend drastically on this parameter.

3. Autocorrelation function

The (normalized) autocorrelation function (AF) of the stationary process x_t, $t = 0, 1, 2, \ldots$ with zero mean value is defined as

$$R(\tau) = \frac{\langle x_t x_{t+\tau}\rangle}{\langle x^2\rangle} . \quad (5)$$

Applying this to the system (1) and (2) one can consider the averaging as the averaging in time, due to ergodicity of the map (2). For usual quasiperiodic motion the AF never reaches 1, although comes very close to 1 at the resonant times (we give a decription of AF and spectra for usual quasiperiodic processes in appendix B).

3.1. Model A: golden mean rotation number

We calculated the AF for the SNA in the system (1) and (2) for the golden mean rotation number $\omega = \omega_{gm}$ and present the results in figure 1. It can be described as follows.

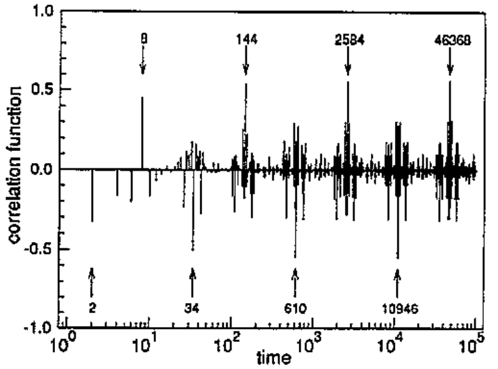


Figure 1. The autocorrelation function for the golden mean rotation number for model A. The main peaks at the resonant times $\tau = q_{3n-1}$ are pointed out.

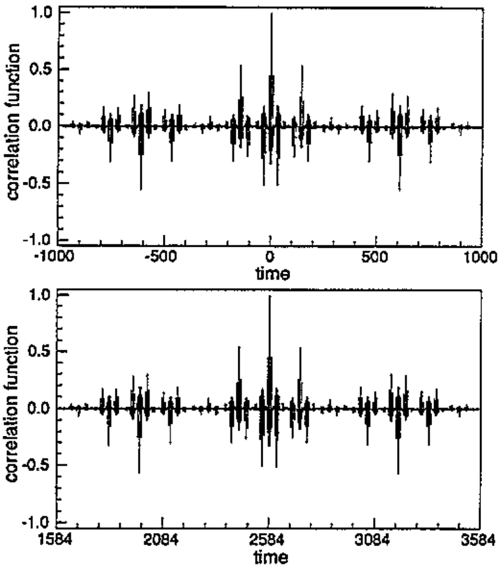


Figure 2. Comparison of two parts of the AF figure 1: one near $\tau = 0$ and the other near $\tau = q_{17} = 2584$ (this part is scaled such that $R(2584) = 1$).

(i) The AF never reaches values close to 1 for $\tau > 0$. We have found that $\max_{\tau\neq 0} |R(\tau)| \approx 0.55$.

(ii) The AF reaches maximum values for $\tau_n = q_{3n-1} = 2, 8, 34, \ldots$, where the 'resonant times' q_n are defined by the expansion of the $\omega_{\rm gm}$ into continuous fraction (see appendix A). Note, that only even resonant times are present. For $n > 3$ the values of the AF at these resonances are approximately constant, only the sign is alternating.

(iii) The AF is extremely small for odd τ. We have found, e.g. that the sum $\sum_{\tau=1}^{500} R^2(2\tau - 1)$ is of order of 10^{-5}, suggesting that one can neglect the values of $R(\tau)$ for odd τ.

(iv) The AF appears to be self-similar. Its structure near the resonant times almost exactly repeats, with appropriate scaling, the structure near $\tau = 0$. This is clearly seen in figure 2, where the region near $\tau = 2584$ is compared to the region near $\tau = 0$. This self-similarity can be quantitatively represented as

$$R(\tau_n \pm \Delta\tau) \approx R(\Delta\tau)R(\tau_n) \tag{6}$$

where $\Delta\tau$ counts significant peaks of resonances at $\tau_0, \tau_1, \ldots, \tau_{n-1}$. We have found that the relation (6) for large n is valid within an accuracy of a few percent, and for many peaks even with much higher accuracy. The whole AF may be thus described as the following.

- The main (zero-order) peaks at τ_n, for large n they have approximately the same amplitude $\overline{R} \approx 0.55$ and alternating signs.
- The first-order peaks at 'harmonics' $\tau_n \pm \tau_k$, $(k < n)$ have amplitudes $R(\tau_n)R(\tau_k) \approx \pm\overline{R}^2$.
- The second-order peaks have amplitudes of order $\overline{R}^3$ at higher-order harmonics, etc.
- There are also peaks at the second harmonics of the times in the hierarchy above, (e.g. at $2\tau_n$) they have relatively small amplitudes (≈ 0.1).

These properties describe the AF of SNA as a self-similar object. The scaling appears as a periodic structure if the AF is presented versus the logarithm of time (figure 1).

3.2. Model A: other rotation numbers

We now describe the properties of AF for some other rotation numbers. 'The second simplest' irrational rotation number is the 'silver mean' $\omega_{sm} = \sqrt{2} - 1$, whose continuous fraction is $[2, 2, \ldots]$ (see appendix A). The AF for the silver mean, presented in figure 3(a), is very similar to that for the golden mean. It can also be described as a self-similar object. The zero-order peaks have slightly larger amplitudes (≈ 0.65), and are placed at the even resonance times $\tau_n = q_{2n} = 2, 12, 70, \ldots$. The values of AF for odd τ are very small, and the second harmonics at $\tau = 2\tau_n$ have relatively high amplitudes. A similar picture is observed for a 'random' irrational number, whose continuous fraction is produced by randomly chosen 1's and 2's (see appendix A). Now zero-order peaks are placed at random times, where the even resonances appear (figure 3(b)), also the alternating of their signs is not perfect.

The observation that the main peaks occur at even resonances suggests investigation of irrationals that do not have such resonances at all. We have studied two examples: modified silver and tin means, described in appendix A. Remarkably, for these irrationals the AF does not have zero-order peaks, only the relatively small peaks at combinational resonances (which, of course, are even) and second harmonics appear (figures 3(c) and 3(d)).

3.3. Models B and C

It is not surprising that in the model B for a non-zero α, when the SNA disappears, the usual structure of quasiperiodic autocorrelation function with many recurrences to one is restored (figure 4(a), note the linear scale of time here in contrast to figure 1). More interesting is the case C, when the SNA still exists, but its AF is similar to that of usual quasiperiodic motion (figure 4(b)). It seems that the structure of quasiperiodic function—whether it is self-similar like in figures 1–3, or resembles usual quasiperiodic behaviour like in figure 4(b)—depends very subtly on the parameter γ. Similar properties have the systems considered in [12, 13], where spectral properties depend subtly on the ratio between parameters. Detailed analysis of this dependence will be presented elsewhere.

4. Spectrum

Usually, two types of power spectrum are observed in dynamical systems: discrete and continuous. Discrete spectrum is represented by δ-peaks at certain frequencies and corresponds to the regular part of the process. Continuous spectrum (often called broad-band noise) corresponds to the irregular component of motion. The spectrum of a regular

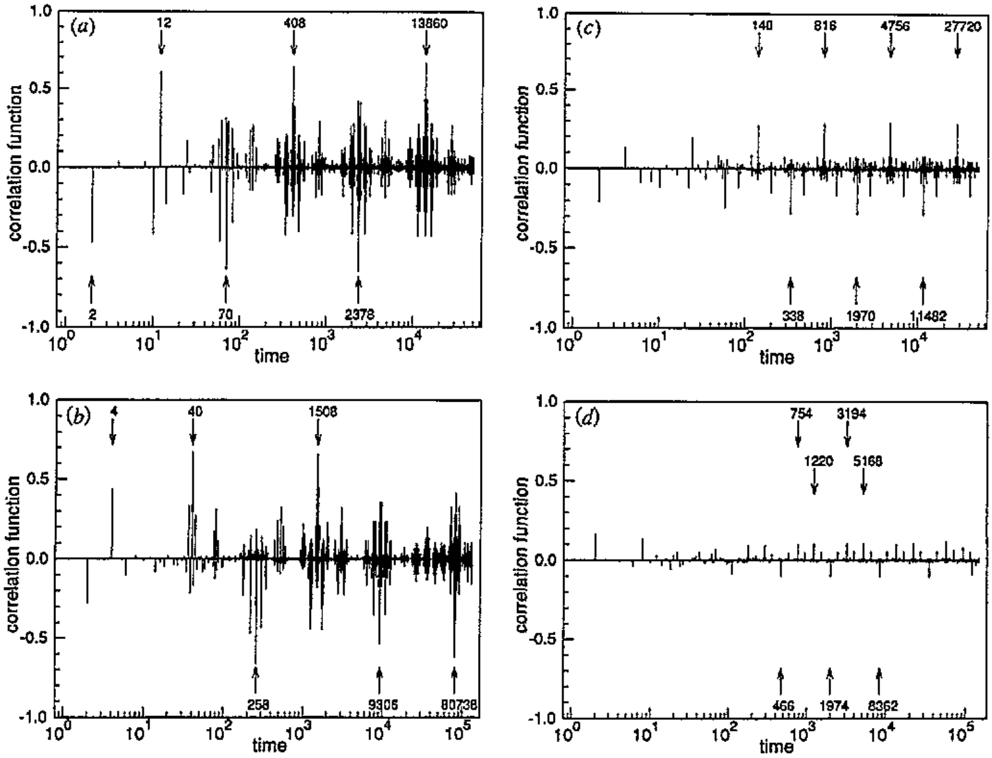


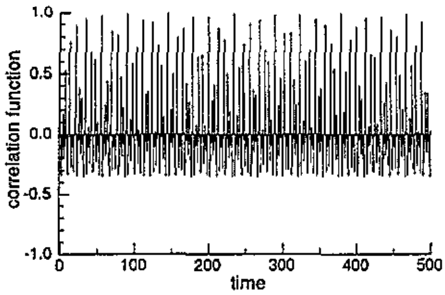
Figure 3. The autocorrelation function for the model A for different rotation numbers: (*a*) silver mean, (*b*) random, (*c*) modified silver mean, (*d*) modified tin mean. For cases (*a*) and (*b*) main even resonances are pointed. For cases (*c*) and (*d*) there are no even resonances. The main peaks pointed in the figures are combinations of resonant times. In the case (*c*): $140 = q_7 - q_5$, $338 = q_8 - q_6$, $816 = q_9 - q_7$, $1970 = q_1 0 - q_8$, etc. In the case (*d*): $466 = q_5 + q_4$, $754 = q_5 \times 2$, $1220 = q_6 - q_5$, $1974 = q_6 + q_5$, $3194 = q_6 \times 2$, etc.

motion (periodic or quasiperiodic) is purely discrete, while chaotic behaviour gives a continuous spectrum (sometimes in combination with a discrete one, e.g. for periodically forced systems).

Recently, a new type of spectrum that is intermediate between discrete and continuous, was described [10–13]; it is called singular continuous spectrum. To define it let us consider a Fourier transform of a process x_k:

$$X(\Omega, T) = \sum_{k=1}^{T} x_k \, e^{i2\pi k\Omega} \tag{7}$$

which defines a path on a plane ($\mathrm{Re}\, X$, $\mathrm{Im}\, X$) when T is considered as time. If in this path there exists a persistent motion (drift), then $|X(\Omega, T)|^2 \sim T^2$ and there is a discrete spectral component at frequency Ω. If the path is random (Brownian motion), then $|X(\Omega, T)|^2 \sim T$ and there is a continuous spectrum for this frequency (if there are both drift and random walk, then the spectrum has both discrete and continuous components). A singular continuous spectral component appears if $|X(\Omega, T)|^2 \sim T^\beta$, where the exponent $\beta \neq 1, 2$. Usually a fractal self-similar path on a plane ($\mathrm{Re}\, X$, $\mathrm{Im}\, X$) corresponds to this component.



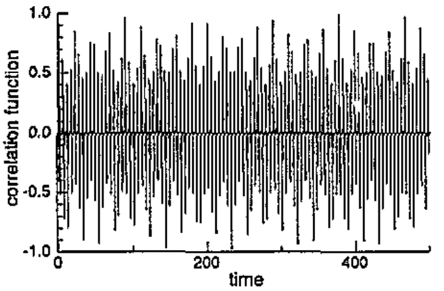


Figure 4. (*a*) The autocorrelation function for the models B with $\alpha = 0.5$. (*b*) The autocorrelation function for the models C with golden mean rotation number and $\gamma = -\cos(\pi\omega)$.

4.1. Model A: golden mean rotation number

In our study of spectral properties we are basing on the ideas of [12]. As it follows from the results of [12], one can expect to obtain non-zero spectral peaks at frequencies $\Omega = (l + m\omega)/n$, with integer l, m, n. In figure 5 we show $|X(\Omega, T)|^2$ versus T for some frequencies Ω. We see that for some frequencies ($\Omega = \frac{1}{4}$ and $\Omega = \frac{1}{4}\omega$) the logarithm of the spectrum appears to grow linearly with $\log T$, with periodic modulation. For other frequencies (e.g. $\Omega = \omega$) the spectrum does not grow. We have tried different frequencies $\Omega = (l + m\omega)/n$, and never found a discrete or a continuous component of the spectrum. The spectrum seems to be purely singular continuous, and different components of it have different exponents β, e.g. $\beta(\Omega = \frac{1}{6}) \approx 1.14$, $\beta(\Omega = \frac{1}{3}) \approx 1.2$, $\beta(\Omega = \frac{1}{4}) = \beta(\Omega = \frac{3}{4}) \approx 1.58$, $\beta(\Omega = \frac{1}{6}\omega) \approx 1.14$, $\beta(\Omega = \frac{1}{4}\omega) \approx 0.98$, while $\beta(\Omega = \frac{1}{2}) = \beta(\Omega = 0) = \beta(\Omega = \frac{1}{2}\omega) = 0$. It appears that to the each of the above mentioned values of Ω a set of frequencies with the same behaviour is attached: $\beta(\Omega + k\omega \bmod 1) = \beta(\Omega)$ for integer k.

The power law behaviour of the 'spectral random walk' corresponds to a nice fractal object on the $(\mathrm{Re}\,X, \mathrm{Im}\,X)$ plane [12]. Graphs at the suitably adjusted times (corresponding to the period of modulation in figure 5) show a self-similar walk (figure 6). Note that these times are exactly the resonant times for which the main peaks in the correlation function appear (in fact, only the ratio of these times is important). It is worth noting that the self-similarity is more perfect for large T (one can also see this from figure 5, where periodicity is not perfect for $T < 200$).

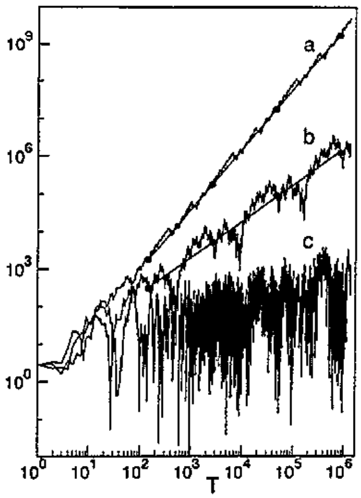


Figure 5. Spectral components $|X(\Omega, T)|^2$ as function of T for the model A with golden mean rotation number. Curve (a) $\Omega = \frac{1}{4}$, curve (b) $\Omega = \frac{1}{4}\omega$, curve (c) $\Omega = \omega$. On the curves (a) and (b) the full circles mark positions $q_{12} = 144$, $q_{18} = 2584$, $q_{24} = 46\,368$, $q_{30} = 832\,040$ which appear to be periods of the spectrum, some of these points are presented in figure 6. The lines correspond to the best linear fits with the slopes given in the text. In the case (c) the spectrum seems not to grow with T (or at least the growth rate is very small).

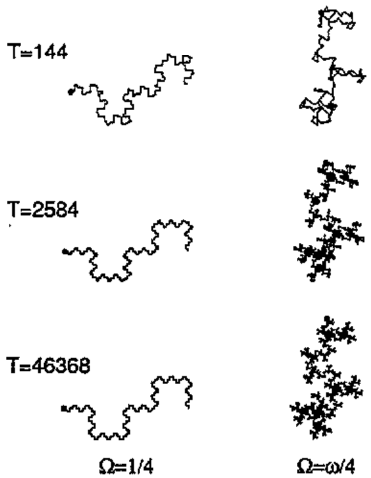


Figure 6. Spectrum of the SNA as a path on the (Re X, Im X) plane, for frequencies $\Omega = \frac{1}{4}$ and $\Omega = \frac{1}{4}\omega$. The pictures for different T are scaled to make self-similar structure of the fractal evident. The dots mark the starting point $T = 0$.

4.2. Model A: other rotation numbers

The properties of the spectrum for quadratic rotation numbers (see appendix A) are very similar to that for the golden mean. There seems to be no continuous and discrete components in the spectrum, non-trivial exponents β can be found for some frequencies, and for these frequencies one can observe a self-similar fractal structure on the (Re X, Im X) plane. Slightly different properties demonstrate the 'random' rotation number. Here again for some frequencies a power-law growth of the spectrum can be observed, but now 'modulation' appears not to be regular, see figure 7(a). The corresponding curve on the (Re X, Im X) plane (figure 7(b)) appears to be a random fractal, with turns left and right not repeating themselves for different scales. For some frequencies (e.g. curve b in figure 7(a)) it is difficult to judge whether there is really a power-law growth of the spectrum, or it saturates.

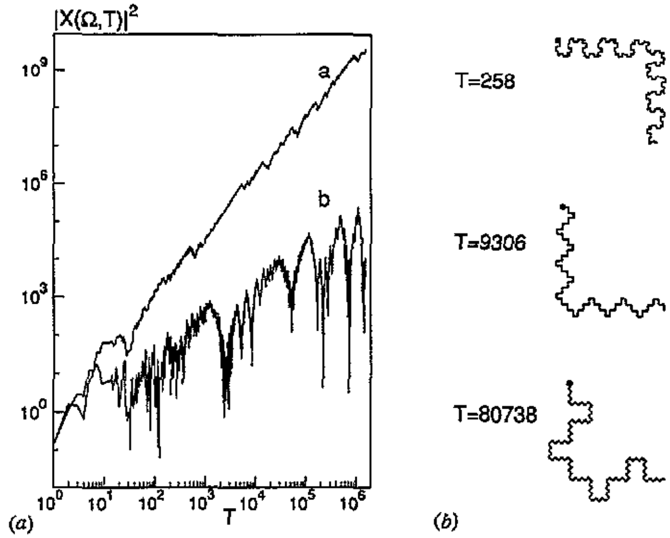


Figure 7. The same as figures 5 and 6, but for a 'random' rotation number (see the discussion in the text).

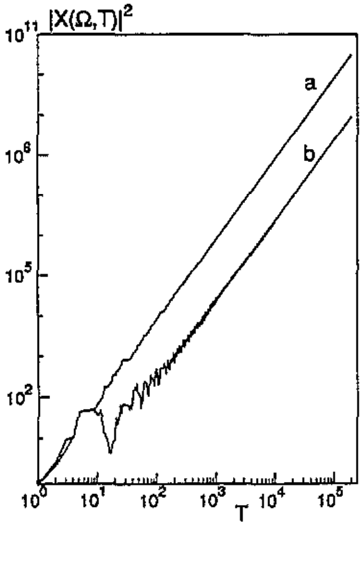


Figure 8. The same as figure 5, but for the model C (parameters are the same as in figure 4(*b*)). Curve (a) $\Omega = (1+\omega)/2$, curve (b) $\Omega = (1+7\omega)/2$. The asymptotic slopes are almost exactly 2.

4.3. Models B and C

If the correlation function is similar to that for usual quasiperiodic motion, we can expect that the spectrum will also be discrete (see appendix B). Indeed, for the model C, whose AF is presented in figure 4, we have found that $\beta = 2$ for the frequencies $\Omega = (1 + (2k-1)\omega)/2 \bmod 1$, k—integer (see figure 8). For this case of purely discrete spectrum we have also applied the approach of [3, 9], where the spectral properties of SNA were discussed. The number of peaks, exceeding a threshold amplitude, indeed scales as a power law, as was suggested in [3]. However, a similar power law can be observed for the model B, where the non-strange torus is rather close to the SNA. Moreover, as one can see from (B1), in the case of a non-strange torus which is discontinuous only in one point, this power law will also occur. At present we can mention only one qualitative difference between spectra of SNA in the model C and spectra of usual quasiperiodic motion: in the latter δ-peaks appear at the frequencies $k\omega$ mod 1, while in the former at the frequencies $(1 + (2k - 1)\omega)/2$ mod 1.

5. Discussion and conclusion

We have demonstrated that the strange non-chaotic attractor may have rather unusual spectral and correlation properties. Its autocorrelation function can have a self-similar structure with peaks of moderate (neither close to one, nor small) amplitude occurring at resonant times of the quasiperiodic forcing. The spectrum is singular continuous and is represented by a fractal curve on the complex plane.

We have considered only the simplest model of SNA. The question thus arises, whether the observed properties are general. First, it should be noted that already within this simple symmetrical model some modifications of forcing (model C above) lead to the disappearance of the singular continuous spectrum, and a discrete spectrum similar to that of a usual quasiperiodic motion (albeit at *unusual* frequencies) is observed. This is probably connected to subtle properties of irrationals, as discussed in [12]. We hope to discuss this point more thoroughly in the future. Second, it is worth noting that the spectrum is not an invariant of the system (in contrast, e.g. to the Lyapunov exponent) and depends on an observable. In general, one can expect that the observable represents a composition of SNA and a circle map (e.g. for system (1) and (2) a general observable $y = F(x, \theta)$ may be a function of both x and θ). In this case a mixture of usual discrete spectrum and a singular continuous spectrum may appear, and at present it is not clear how they can be separated.

One conclusion that can be made from the results of our paper is that usual methods of determining the power spectrum may be inadequate for SNA. Indeed, in these methods (see, for example, [15]) it is implicitly supposed that all spectral components are of the same nature (discrete or continuous), therefore singular continuous components are not detectable. Only in the case when there is no singular continuous spectrum (model C), standard methods may be applied, and then the question of distinguishing between spectra of SNA and usual quasiperiodic attractor makes sense. In [3, 4] it was proposed to measure how many spectral components exceed some threshold. In the case of SNA this number decreases with the threshold value as a power law. However, as it follows from the expressions for the power spectrum of usual quasiperiodic motion, if the observable is a *discontinuous* function of the angle variable, then such a power law will be observed as well (see (B1)).

Finally, in our analysis of the power spectrum we mainly used the approach of [12]. It happened that the spectral properties of the SNA considered above are rather close to the properties of a discrete symbolic model considered in [12]. In particular, our figure 6 is similar to figure 6 in [12]. Therefore, we hope that an appropriate symbolic model will allow us to explain the correlation and spectral properties of SNA [14].

Acknowledgments

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Appendix A. Basic properties of Diophantine approximations [16, 17]

Each irrational $0 < \omega < 1$ can be represented as an infinite continuous fraction

$$\omega = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}} = [a_1, a_2, \ldots] \tag{A1}$$

with integer a_i. The best rational approximations for the irrational are obtained if one takes finite continuous fractions

$$\omega_n = \frac{p_n}{q_n} \tag{A2}$$

where p_n and q_n satisfy the recursion relations

$$p_n = a_n p_{n-1} + p_{n-2} \qquad q_n = a_n q_{n-1} + q_{n-2} \qquad p_0 = 0 \qquad q_0 = p_1 = 1 \qquad q_1 = a_1 . \tag{A3}$$

At the 'resonant times' q_n the circle map (2) nearly repeats itself, because $|\omega - p_n/q_n| < 5^{-1/2} q_n^{-2}$ (e.g. for the golden mean $|\omega_{\mathrm{gm}} - p_n/q_n| = q_n^{-1}(\sqrt{5} q_n + (-\omega_{\mathrm{gm}})^{n+1})^{-1}$).

The quadratic irrationals (solutions of the quadratic equation with integer coefficients) are represented by periodic continuous fractions. Below we present continuous fractions for some irrationals, used in this paper.

Golden mean: $\omega_{\mathrm{gm}} = (\sqrt{5} - 1)/2 = [1, 1, 1, \ldots]$

$$p_n/q_n = \tfrac{1}{1}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{5}, \tfrac{5}{8}, \tfrac{8}{13}, \tfrac{13}{21}, \tfrac{21}{34}, \tfrac{34}{55}, \tfrac{55}{89}, \tfrac{89}{144}, \tfrac{144}{233}, \tfrac{233}{377}, \tfrac{377}{610}, \tfrac{610}{987}, \tfrac{987}{1597}, \ldots$$

Silver mean: $\omega_{\mathrm{sm}} = \sqrt{2} - 1 = [2, 2, 2, 2, \ldots]$

$$p_n/q_n = \tfrac{1}{2}, \tfrac{2}{5}, \tfrac{5}{12}, \tfrac{12}{29}, \tfrac{29}{70}, \tfrac{70}{169}, \tfrac{169}{408}, \tfrac{408}{985}, \tfrac{985}{2378}, \tfrac{2378}{5741}, \ldots$$

Bronze mean: $\omega = (\sqrt{13} - 1)/2 = [3, 3, 3, \ldots]$

$$p_n/q_n = \tfrac{1}{3}, \tfrac{3}{10}, \tfrac{10}{33}, \tfrac{33}{109}, \tfrac{109}{360}, \tfrac{360}{1189}, \tfrac{1198}{3927}, \ldots$$

Tin mean: $\omega = \sqrt{5} - 2 = [4, 4, 4, 4, \ldots]$

$$p_n/q_n = \tfrac{1}{4}, \tfrac{4}{17}, \tfrac{17}{72}, \tfrac{72}{305}, \tfrac{305}{1292}, \tfrac{1292}{5473}, \ldots$$

Modified silver mean: $\omega = \sqrt{2}/2 = [1, 2, 2, 2, \ldots]$

$$p_n/q_n = \tfrac{1}{1}, \tfrac{2}{3}, \tfrac{5}{7}, \tfrac{12}{17}, \tfrac{29}{41}, \tfrac{70}{99}, \tfrac{169}{239}, \tfrac{408}{577}, \tfrac{985}{1393}, \ldots$$

Modified tin mean: $\omega = (\sqrt{5} + 1)/4 = [1, 4, 4, 4, \ldots]$

$$p_n/q_n = \tfrac{1}{1}, \tfrac{4}{5}, \tfrac{17}{21}, \tfrac{72}{89}, \tfrac{305}{377}, \tfrac{1292}{1597}, \ldots$$

'Random irrational':

$$\omega = [1, 2, 1, 1, 1, 2, 1, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2, 1, \ldots]$$

$$p_n/q_n = \tfrac{1}{1}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{5}{7}, \tfrac{8}{11}, \tfrac{21}{29}, \tfrac{29}{40}, \tfrac{79}{109}, \tfrac{187}{258}, \tfrac{453}{625}, \tfrac{640}{883}, \tfrac{1093}{1508}, \tfrac{2826}{3899}, \tfrac{6745}{9306}, \ldots$$

Appendix B. Correlations and spectra of usual quasiperiodic motion

Suppose that the observed variable y_n may be represented as a one-valued function of the circle variable θ_n as $y = G(\theta)$, and θ_n is governed by the circle mapping (2). The Fourier series of the periodic function $G(\theta)$ is

$$G(\theta) = \sum_{-\infty}^{\infty} g_k \, e^{i2\pi k\theta} \qquad g_{-k} = g_k^* \, .$$

The autocorrelation function is then

$$R(\tau) = \langle y_0 y_\tau \rangle = \sum_{-\infty}^{\infty} |g_k|^2 \, e^{-i2\pi k\tau\omega}$$

itself a quasiperiodic function. It has values close to $R(0)$ at the resonance times $\tau = q_n$ for which $\omega\tau \approx 0$ (mod 1).

The spectrum of y_n

$$Y(\Omega) = \sum_{-\infty}^{\infty} y_n \, e^{i2\pi\Omega n}$$

has only δ-peaks on the harmonics of ω [18]:

$$\langle |Y(\Omega)|^2 \rangle = \sum_{-\infty}^{\infty} |g_k|^2 \delta(\Omega + k\omega) \, . \qquad \text{(B1)}$$

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