Comment on "Noisy Uncoupled Chaotic Map Ensembles Violate the Law of Large Numbers"

Recently Sinha [1] has claimed that an ensemble of uncoupled chaotic maps with spatially uniform parametric fluctuations violates the law of large numbers [case (ii)] in [1], while it is valid for spatially nonuniform fluctuations [cases (iii) and (iv)]. The result is based on the calculations of the mean field and its variance. The numerical results presented in Ref. [1] show that the variance does not decrease as $1/N$, where N is a number of elements in an ensemble, but saturates after some N_c.

In this Comment I show that the law of large numbers is valid for uncoupled chaotic maps [case (ii) in [1]], but should be implemented carefully.

Consider an ensemble of one-dimensional chaotic maps

$$x_{t+1}(i) = f(x_t(i), a_t),$$

where $i = 1, ..., N$ is the site index, $t = 1, 2, 3, ...$ is discrete time, and $a_t = a(1 + \sigma\eta_t)$ is a parameter which is the same for all elements in the ensemble, but fluctuates in time with amplitude σ (η_t is a random number distributed in the interval $[-0.5, 0.5]$). For given initial conditions average field at time t is defined as

$$h_t^{(N)} = \frac{1}{N}\sum_{j=1}^{N} x_t(j).$$

Statistical properties of $h_t^{(N)}$ at the limit $N \to \infty$ are of interest. The variance of $h_t^{(N)}$, defined as

$$v_t^N = \langle (h_t^{(N)} - \langle h_t^{(N)} \rangle_{\rm ic})^2 \rangle_{\rm ic}, \qquad (1)$$

is expected, according to the law of large numbers, to decrease as $1/N$. In (1) $\langle\,\rangle_{\rm ic}$ means averaging over random initial conditions, but the parameter sequence a_t is the same for all of them. In Fig. 1 v is plotted vs N for a tent map $f(x, a) = 1 - a|x|$, $a = 1.8$, $\sigma = 0.1$. No violations of the law of large numbers can be seen, in contrast to Ref. [1]: data do not deviate from a line with slope -1.

Behavior of ensembles of uncoupled maps governed by spatially homogeneous noise was considered in Refs. [2,3]. The ensemble may be described with probability density $W_t(x)$ which satisfies the Perron-Frobenius equation

$$W_{t+1}(x) = \int dy\, \delta(x - f(y, a_t)) W_t(y). \qquad (2)$$

Because the Perron-Frobenius operator explicitly depends on time (via time dependence of parameter a) the probability density function $W_t(x)$ also depends on t. For a fixed time t,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} x_t(j) = \int x W_t(x) dx \equiv \rho_t,$$

and convergence of the limit is in accordance with the law of large numbers, as was shown above.

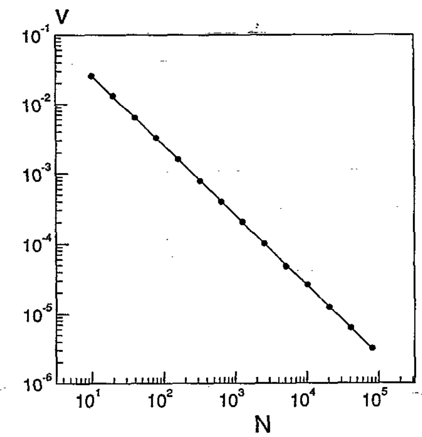

FIG. 1. Variance vs ensemble size for an ensemble of tent maps with $a = 1.8$, $\sigma = 0.1$. The solid line with slope 0.999 gives the best linear fit.

Suppose now that when calculating the average (1), one takes not only different initial conditions, but also different parameter sequences a_t. This means additional averaging over time and gives a quantity

$$V^{(N)} = \langle (h_t^{(N)} - \langle h_t^{(N)} \rangle_{{\rm ic},t})^2 \rangle_{{\rm ic},t}.$$

Taking into account that $\lim_{N\to\infty} \langle h_t^{(N)} \rangle_t = \langle \rho_t \rangle_t$ we conclude that for large N the quantity $V^{(N)}$ does not decrease but saturates to a value $\langle (\rho_t - \langle \rho_t \rangle_t)^2 \rangle_t$. Probably, just this quantity was calculated in Ref. [1]; it can be obtained from the solution of Eq. (2).

In conclusion, I have shown that careful examination of an averaging procedure in an ensemble of uncoupled maps gives results which agree with the law of large numbers, and also allows us to explain the numerical results of Ref. [1]. It seems suggestive to use this approach in the study of globally coupled maps [4].

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