Anomalous Diffusion in the Kuramoto-Sivashinsky Equation

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We study the motion of advected particles in the Kuramoto-Sivashinsky equation. We give numerical evidence as well as analytical arguments for anomalous diffusion in which the particle displacement Δr satisfies $\langle[\Delta r(t)]^2\rangle \sim t^\eta$ where $\eta > 1$. We show that if the flow is initially seeded with many particles, they will coalesce in time and that a passive scalar density tends to an asymptotic (time dependent) distribution, which, for given initial conditions on the velocity field, is independent of the initial distribution of the passive scalar.

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The problem of understanding the motion of advected particles in a turbulent fluid is of both fundamental and practical importance. Unfortunately it is also a very hard problem of which our knowledge is far from complete [1,2]. In the following we shall describe some new results obtained in a simplified variant: fluid flow in one dimension. More precisely, we shall describe the advection of a passive scalar in a simple nonlinear equation for a velocity field—the so-called Kuramoto-Sivashinsky (or Kuramoto-Sivashinsky-Burgers) equation [3,4], which is, perhaps, the simplest nonlinear field equation exhibiting turbulence (i.e., spatiotemporal chaos) and applicable to a wide class of phenomena from chemical turbulence [3] to flame fronts [4]. When the motion of a single "dust particle," immersed in this one-dimensional turbulent "fluid," is followed, we find *anomalous diffusion* [5–10], i.e., the mean squared distance $\langle[\Delta r(t)]^2\rangle$ traversed in time t grows as

$$\langle[\Delta r(t)]^2\rangle \sim t^\eta , \tag{1}$$

where η is *larger than unity.*

Similar effects are seen in the phase space of low-dimensional chaotic systems [11] and experimentally in the spreading of dye of a turbulent fluid surface [12]. Related phenomena have been seen in fluids with ordered roll structures, as, e.g., convective rolls where an exponent $\eta = \frac{1}{3}$ was theoretically predicted [13] and found experimentally [14], and in Richardson's famous "$\frac{4}{3}$ law" [15] for *relative* diffusion is fully developed turbulence, which corresponds to $\eta = 3$. Further one should note that such anomalies are to some extent expected in low-dimensional systems, being the analogs of the "long time tails" seen in higher-dimensional transport coefficients (see, e.g., [16]). Anomalous diffusion has also been studied in relation to particle motion in models of 2D turbulence [17], when an anomalous power spectrum for the stream function is assumed, and by renormalization group methods (in arbitrary dimension) by assuming different forms of the subgrid velocities [18].

In the Kuramoto-Sivashinsky (KS) equation we find that particles *coalesce* instead of drifting apart, but the motion of a single particle shows the anomalous exponent. The effect seems to be intimately connected with the *cellular structure* being caused by large scale motions of the pattern as a whole. As we shall see the strongly irreversible character of the motion of random walkers (Fig. 4) is a powerful probe of the underlying dynamics.

The equations we study are the following:

$$\frac{\partial u}{\partial t} + u\nabla u = -\nabla^2 u - \nabla^4 u , \tag{2}$$

$$\frac{dr}{dt} = u(r(t),t) , \tag{3}$$

where $u(x,t)$ is a velocity field in one dimension and $r(t)$ is the trajectory of a particle immersed in the fluid. The velocity field becomes turbulent with an irregular cellular structure and the particle is attracted to the points where the (strongly compressible) fluid converges, i.e., the zeros in the velocity field, where u is positive to the left and negative to the right. The other zeros are repelling.

We have initiated (2) with a random initial condition (and periodic boundary conditions on a system of length L) and let them evolve in time until a statistically steady turbulent state is obtained [19,20]. Then we start (3) at some random position and study the subsequent evolution of $r(t)$ and u. Although the motion appears random, the particle moves almost ballistically for long stretches—of the order of several thousands of time units in a system of $L = 1000$. This anomalous appearance can be quantified by looking at the mean square of the traversed distance $\langle[\Delta r(t)]^2\rangle$ which is defined by averaging the square of the displacement $[\Delta r(t)]^2 = [r(t'+t) - r(t')]^2$ over the starting time t'. In Fig. 1 a typical plot of $\langle[\Delta r(t)]^2\rangle$ is shown. It was obtained by averaging over t' in (several) histories $r(t)$ of 50000 time units in a system of effective size $L = 4277.5$ (using a spectral method [19]). It is seen that, for times t up to around 10000, we can fit very well by the form (1) with an exponent $\eta \simeq 1.4$. For the largest times normal diffusion is regained.

We have simulated the Kuramoto-Sivashinsky equa-

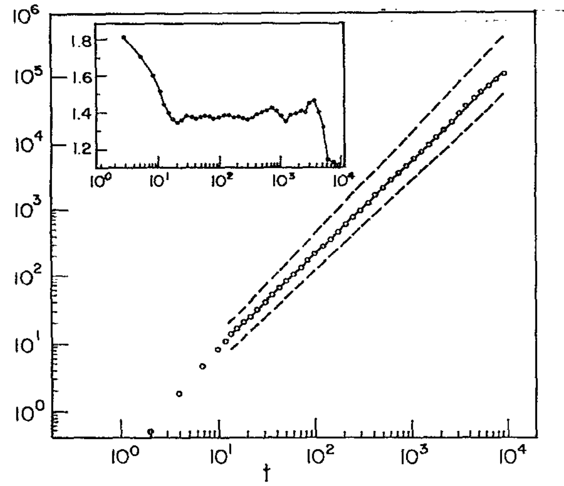

FIG. 1. $\langle[\Delta r(t)]^2\rangle$ vs t. The two dashed lines have slopes $\frac{3}{2}$ and $\frac{4}{3}$, respectively. The best fit gives $\eta=1.38$. The inset shows the logarithmic derivative.

tion in different ways: by simple finite differencing with different spatial resolution (lattice spacing 1 and 0.5) and time steps (0.1 and 0.05) and further by a spectral method [19]. In all cases the diffusion is clearly anomalous, in the $L=1000$ case for times up to a few thousand. The value of the exponent η varies, however, in the range $\frac{4}{3}<\eta<\frac{3}{2}$. For a system size $L=1000$ and lattice spacing 1 we thus find $\eta=1.33$, whereas we find $\eta=1.45$ for lattice spacing 0.5 (which is as small as we can make it since every doubling of the spatial resolution costs a factor 16 in computing time). For smaller systems we see similar behavior, but the crossover happens at shorter times. We have also looked at higher moments of $\Delta r(t)$, with moments p up to 8, and we find that the scaling is characterized very well by *one* scaling exponent independent of p. Thus $\langle|\Delta r(t)|^2\rangle \sim t^{\eta p/2}$ and there is no sign of "multifractality."

To understand these results from an analytic point of view, we shall use the well-known equivalence between the Kuramoto-Sivashinsky equation and the Burgers equation with *noise* [19–23]. In the latter equation

$$\frac{\partial u}{\partial t}+u\nabla u=\nu\nabla^2 u+\eta(x,t)\,, \tag{4}$$

the instability supplied by the negative surface tension in (2) has been replaced by the noise η, where $\langle\eta(x,t)\times\eta(x',t')\rangle=-\Gamma\nabla^2\delta(x-x')\delta(t-t')$ and a *positive* surface tension.

For short times and small systems the statistical properties of (4) can be found from the linearized version [24]

$$\frac{\partial u}{\partial t}=\nu\nabla^2 u+\eta(x,t)\,. \tag{5}$$

For this equation it is easy to compute correlation functions. The general two-point correlation function for (5) is

$$\langle u(x+r,t+\tau)u(x,t)\rangle\to\frac{\Gamma}{4(\pi\nu^3\tau)^{1/2}}e^{-r^2/4\nu\tau}\,, \tag{6}$$

where the limit means $t\to\infty$ (to get a stationary process) and infinite system size, $L\to\infty$.

In general, if we consider the Langevin equation $\dot r=f(t)$ we can express the diffusion in terms of the correlation function for f by the so-called Green-Kubo formula (see, e.g., [16]):

$$\langle[r(t)-r(0)]^2\rangle\to 2t\int_0^t\langle f(\tau)f(0)\rangle d\tau\,, \tag{7}$$

where again t must be large enough for the process to be stationary. Now, in the equation for the advected particle, $f(t)=u(r(t),t)$ which we do not know until $r(t)$ has been calculated. To overcome this difficulty in a self-consistent way, we shall now make the bold assumption that we can describe the statistics of the advected particle by the Green-Kubo formula, where the correlation function $\langle f(\tau)f(0)\rangle$ is replaced by $\langle u(x+r,t+\tau)u(x,t)\rangle$ evaluated at $\Delta r=\Delta r(\tau)=\{\langle[r(\tau)-r(0)]^2\rangle\}^{1/2}$. This is a kind of *mean field* approach akin to those used in classical turbulence theory [2]. Although this approximation will certainly get numerical factors wrong, we hope that the scaling exponent η will be correctly reproduced.

We first apply this approximation to the *linearized* equation (5) together with (3). If we let $[\Delta r(t)]^2=4\nu t\times\phi(t)$ we get

$$\phi(t)=\frac{1}{t_0^{1/2}}\int_0^t\frac{1}{\sqrt{2\tau}}e^{-\phi(\tau)}d\tau\,, \tag{8}$$

where [25] $t_0=16\pi\nu^6/\Gamma^2$. The solution is

$$[\Delta r(t)]^2=4\nu t\phi(t)=4\nu t\ln[1+(t/t_0)^{1/2}]\,, \tag{9}$$

which means that the diffusion is *anomalous* with exponent $\eta=\frac{3}{2}$ up to time t_0 and then crosses over to the behavior

$$\langle[r(t)-r(0)]^2\rangle\sim t\ln t\,. \tag{10}$$

Unexpectedly, it never becomes quite normal. Such logarithmic corrections have been predicted earlier in the context to fluid flows through porous media [26]. For finite systems the diffusion does finally become normal. The largest correction to (6) in the limit $L\gg 1$ is $L^{-1}\times\exp[-(2\pi/L)^2\nu t]$ (coming from the replacement of the discrete sum over Fourier components $k_n=2\pi n/L$ by an integral) and will lead to normal diffusion for times larger than $t_1=L^2/4\pi^2\nu$.

For the nonlinear case (4) the precise form of the correlation function is not known. Only the scaling form

$$\langle u(x+r,t+\tau)u(x,t)\rangle\to\tau^{-1/z}H(r^z/\tau)\,, \tag{11}$$

where the scaling function $H(\xi)\to 0$ for $\xi\to\infty$ and $H(\xi)\to$ const for $\xi\to 0$, and where the dynamical exponent $z=\frac{3}{2}$ [21]. Applying the same mean field approach as above we now see that there exists a self-consistent solution

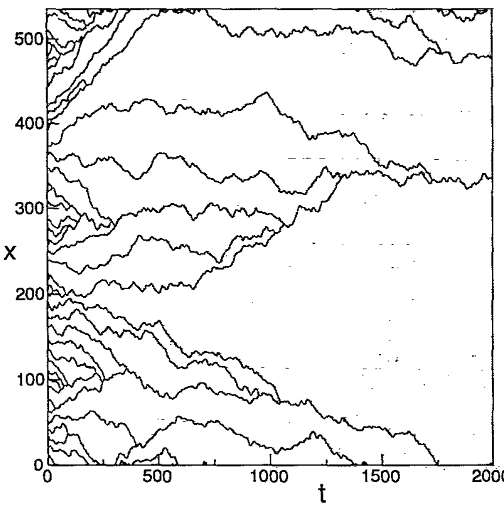

FIG. 2. Space-time plot of many walkers. The flow was initially seeded with 46 walkers.

$$\langle [r(t)-r(0)]^2 \rangle \sim t^{4/3}, \tag{12}$$

even asymptotically, in contrast to the case $z=2$ treated above. We thus expect the diffusion to be anomalous in the nonlinear case, cut off only by finite system effects. It is interesting that asymptotic power law behavior is only possible when $z=\frac{3}{2}$.

For the Kuramoto-Sivashinsky equation the true asymptotic properties should be given by the nonlinear case above with the result $\eta=\frac{4}{3}$. We know, however, that (2) behaves like the linear system for small L and that the crossover to (4) is very slow [19,20]. Further we know that the effective surface tension ν is very large, of the order of 10, due to the cellular structure [20], which means that t_0 defined above is huge. Finally we know, also from [20], that the precise way in which we simulate (2) changes the effective parameters in (4) and thus changes the crossover point. Thus we should in principle be able to see both $t^{3/2}$, $t\ln t$, t, and $t^{4/3}$ depending on system size and effective parameters. Of course we cannot rule out the possibility that the asymptotic value of η is different from $\frac{4}{3}$ since we do not know the accuracy of the mean field approximation made.

If we seed the flow with several particles, again propagated by (2) and one (3) for each particle we find that they *coalesce* in time. Figure 2 shows the fate of 46 particles. After 2000 iterates only 5 remain. In this plot we do indeed see some indication that the long stretches of almost ballistic motion are caused by coherent motion of patches in the cellular structure, since nearby walkers tend to follow each other. This can also be seen in Fig. 3 where the zeros of the velocity field are shown. The thick lines are the "attractive" zeros, where the particles like to sit, whereas the thin lines are the repelling zeros. We see that *attractive lines do not split* in time. They only

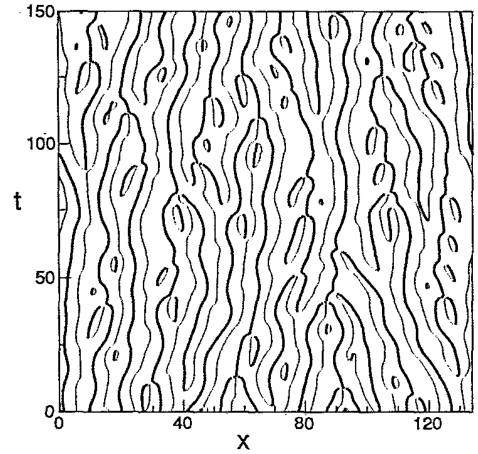

FIG. 3. Space-time plot of the zeros of the velocity field.

coalesce, and pairs of attractive/repelling lines can be created. The splitting, on the other hand, takes place around the unstable zeros. This asymmetry is related to the invariance of (2) under $u \to -u$, $x \to -x$, $t \to -t$; i.e., attractive lines can be changed to repelling lines *backwards in time*. If we think of Fig. 3 as a small part of a long time history, the walkers will populate some of the attractive lines emerging at the bottom of the picture. Subsequently these lines meander and coalesce with other attractive lines, but do not split. Thus the walkers will diffuse and coalesce as well. Figure 2 is extremely reminiscent of paths formed by directed polymers in a random environment [27]. In fact, the scaling exponent $\eta=\frac{4}{3}$ is precisely the one found there.

The "attracting," i.e., coalescing, property of the many particle system can be seen clearly when we study the properties of a passively advected scalar field—i.e., a density of particles. The equation for a passive scalar, conserved density T is (see, e.g., [28])

$$\frac{\partial T}{\partial t}+\nabla(Tv)=\kappa\nabla^2 T, \tag{13}$$

combined with the KS equation (2). What we have studied above is the limit $\kappa \to 0$.

In Fig. 4 we show the field T after 10^5 times units (for $L=1000$ and $\kappa=2$). The initial condition was $T=0$ everywhere except at $x=500$, where it was 1, i.e., a "single drop of ink" in the center. Again, the final density clearly reflects the cellular structure. And, strangely enough at first sight, the distribution is *unique*: Although it will still vary in time, different initial distributions $T(x)$ will all lead to the same final state—provided, naturally, that the initial conditions for the velocity field are the same. Thus Fig. 4 could be obtained just as well with a uniform initial $T(x)=$const. It is amazing that the dynamics of the passive scalar retains this attracting property even though the velocity field is turbulent in the sense of having many positive Lyapunov exponents (see, e.g.,

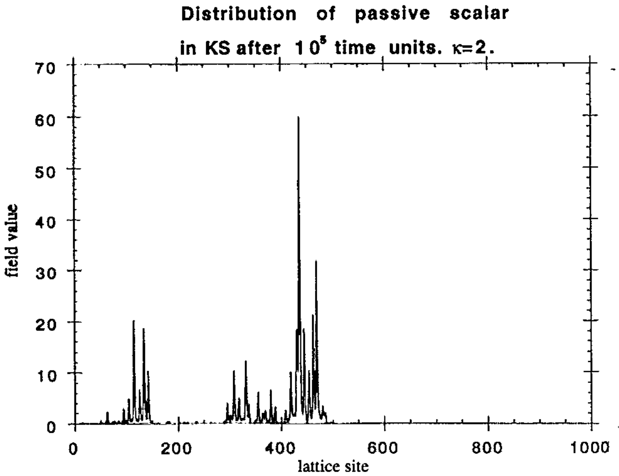

FIG. 4. The passive scalar field T, evolved from a single "drop of ink in the center."

[29]). It would be of importance to understand the validity of the "mean field" approximations used here to calculate the exponent η. Two recent papers analyze properties of passive scalar motion by path integrals [30] and by renormalization group flows [31]. Maybe these techniques can be extended to the problem treated here.

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