Physics Letters A 181 (1993) 149–152
North-Holland

PHYSICS LETTERS A

Multistep method for controlling chaos

Igor M. Starobinets [a] and Arkady S. Pikovsky [a,b]

[a] *Applied Physics Institute, Russian Academy of Sciences, Nizhni Novgorod, Russian Federation*
[b] *Max-Planck-Arbeitsgruppe "Nichtlineare Dynamik", University of Potsdam, Potsdam, Germany*

Received 1 February 1993; accepted for publication 6 August 1993
Communicated by D.D. Holm

We present a multistep control method where the trajectory on a chaotic attractor is directed by small perturbations towards a chosen fixed point. The method gives a significant reduction of the chaotic transient preceding the controlled motion as compared with the Ott–Grebogi–Yorke method. Transition from local to global control, when chaotic transients disappear, is discussed.

Recently Ott, Grebogi and Yorke proposed a method for controlling chaos [1]. They showed how with small external perturbations it is possible to change the operating regime of a system with chaotic behaviour, in particular, how to stabilize one of the unstable fixed points within the chaotic attractor. Since then this method was realized experimentally [2], and was further developed theoretically [3]. For other methods of controlling chaos see refs. [4,5].

An important feature of the Ott–Grebogi–Yorke (OGY) method is that it uses linear equations near the fixed point chosen for control. Thus, if the system is far away from this fixed point, control is not applied. The global strategy of the OGY approach is the following: first wait until the system comes in a small neighborhood of the fixed point, and then activate control. Thus, one of the main characteristics of control – mean time to achieve control starting from a randomly chosen initial point – is in fact the mean time of falling of the trajectory in a small neighborhood of the fixed point in the unperturbed system, this time is defined by the ergodic properties of the chaotic attractor [1].

Thus, it seems reasonable to generalize the OGY method, going beyond the linear approximation near the fixed point and trying to control the system even at large distances from the desired state. Such a generalization is proposed in this paper. We shall show that this method significantly reduces the mean time necessary to achieve control.

To simplify the analysis we shall describe the method as applied to the two-dimensional map $\boldsymbol{F}$ that contains a chaotic attractor. Let the system depend on some control parameter $u \in (-u_*, u_*)$ with maximal possible perturbation u_*. Usually, u_* is small so we can consider the perturbation as linear and write the mapping in the form

$$\boldsymbol{x}(t+1)=\boldsymbol{F}(\boldsymbol{x}(t))+\boldsymbol{b}(\boldsymbol{x}(t))u\,. \tag{1}$$

Let $\boldsymbol{x}_F$ denote an unstable fixed point on the attractor. The objective of control is to stabilize this fixed point using some feedback control law. In a small neighborhood of $\boldsymbol{x}_F$ the OGY method ensures such a stabilization. We, however, want to control the system even when it is far from the fixed point.

Suppose we start form a point on the attractor $\boldsymbol{x}(0)$. Let us fist fix the number of time steps, N, and try to find such a sequence of control parameters

$$u(0), u(1), ..., u(N-1)\,,$$

that minimizes the distance between $\boldsymbol{x}(N)$ and the fixed point $\boldsymbol{x}_F$. So we have a minimization problem

$$[\boldsymbol{x}(N)-\boldsymbol{x}_F]^2 \overset{!}{=} \min\,, \tag{2}$$

as the equation for finding $u(0)$, ..., $u(N-1)$. The final state $\boldsymbol{x}(N)$ depends on the control parameters according to (1), and solving (2) is a nontrivial problem. We can simplify it assuming that the control parameters are very small and by linearizing near the unperturbed trajectory $x_0(0), ..., x_0(N)$. In this approximation we can write $x(N)$ as

$$x(N)=x_0(N)+\sum_{j=0}^{N-1} \hat{W}(j)b(x_0(j))u(j)\,, \qquad (3)$$

where

$$\hat{W}(j)=\frac{\partial x_0(N)}{\partial x_0(j+1)}=\prod_{i=j+1}^{N-1}\frac{\partial F(x_0(i))}{\partial x}\,.$$

Substituting (3) in (2) we get

$$\left(x_0(N)-x_F+\sum_{j=0}^{N-1}\hat{W}(j)b(x_0(j))u(j)\right)^2 \overset{!}{=} \min\,. \qquad (4)$$

This condition defines a $(N-1)$-dimensional hyperplane in the space $(u(0), ..., u(N-1))$. The solution, however, does not lie on this hyperplane because of the restriction $|u(j)|\leqslant u_*$. Thus, the control sequence should be defined, as it usually is done in the theory of optimal control [6], as follows,

$$\begin{aligned} u(j)&=-\mathrm{sign}(r)u_*\,, \quad &&\text{if } r>1\,,\\ &=-r\,, &&\text{if } r\leqslant 1\,, \end{aligned} \qquad (5)$$

where

$$r=\frac{[x_0(N)-x_F]\hat{W}(j)b(x_0(j))}{|\hat{W}(j)b(x_0(j))|^2}\,.$$

Let us analyze the control feedback law (5) in the case $N=1$ (one-step control). Suppose that the system is very near to the fixed point x_F, so that $r<1$. Then from (1), (5) we get

$$x(t+1)=F(x(t))-b(x(t))\frac{[F(x(t))-x_F]b(x(t))}{|b(x(t))|^2}\,. \qquad (6)$$

We can linearize near x_F to get

$$y(t+1)=\hat{A}y(t)-\frac{B(\hat{A}y(t))B}{|B|^2}\,, \qquad (7)$$

where

$$\hat{A}=\frac{\partial F(x_F)}{\partial x}=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},$$

$$B=b(x_F)=(b_1,b_2)\,.$$

This is a linear mapping with eigenvalues easily obtained as

$$\lambda_1=0\,,$$
$$\lambda_2=[b_2^2a_{11}+b_1^2a_{22}-b_1b_2(a_{12}+a_{21})](b_1^2+b_2^2)\,. \qquad (8)$$

For arbitrary b we cannot expect that $|\lambda_2|<1$, so this one-step method does not ensure control, in contrast to OGY [1]. Nevertheless, we may try to apply the N-step control in order to reach the neighborhood of the fixed point x_F, and in this neighborhood either apply the one-step method if $|\lambda_2|<1$, or the OGY method, which ensures control for almost all b.

Thus, our procedure is as follows. Given an initial point $x(0)$, we try to apply control feedback with $N=1, 2, ..., N_{max}$. For each N we calculate the control sequence form (5), then substitute this sequence in (1) and calculate $x(N)$. If the condition

$$|x(N)-x_F|<\epsilon\,, \qquad (9)$$

is satisfied with small ϵ, then this value of N gives successful control and we may apply it, using after at the final stage the one-step or the OGY method. If condition (9) is not satisfied, we increase N by 1 and try again. If condition (9) cannot be fulfilled for all $1\leqslant N\leqslant N_{max}$, then we do not apply control at all and make an iteration with $u(0)=0$. After this, the whole procedure is repeated.

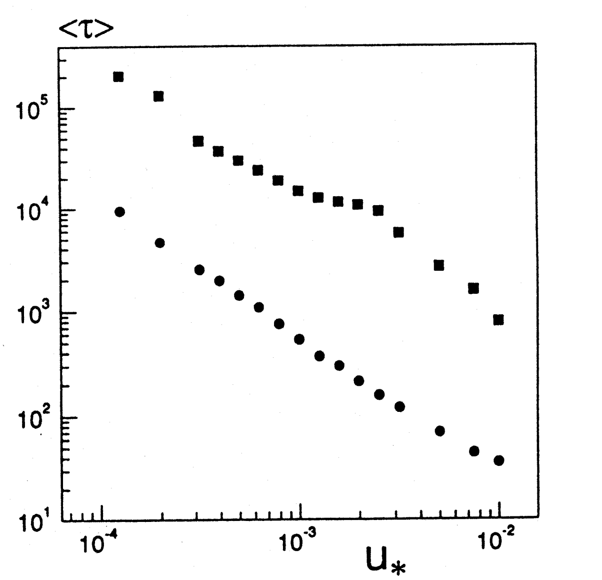

Fig. 1. $\langle\tau\rangle$ versus u_*. Circles: multistep method with $N_{max}=30$, $\epsilon=u_*$; squares: OGY method.

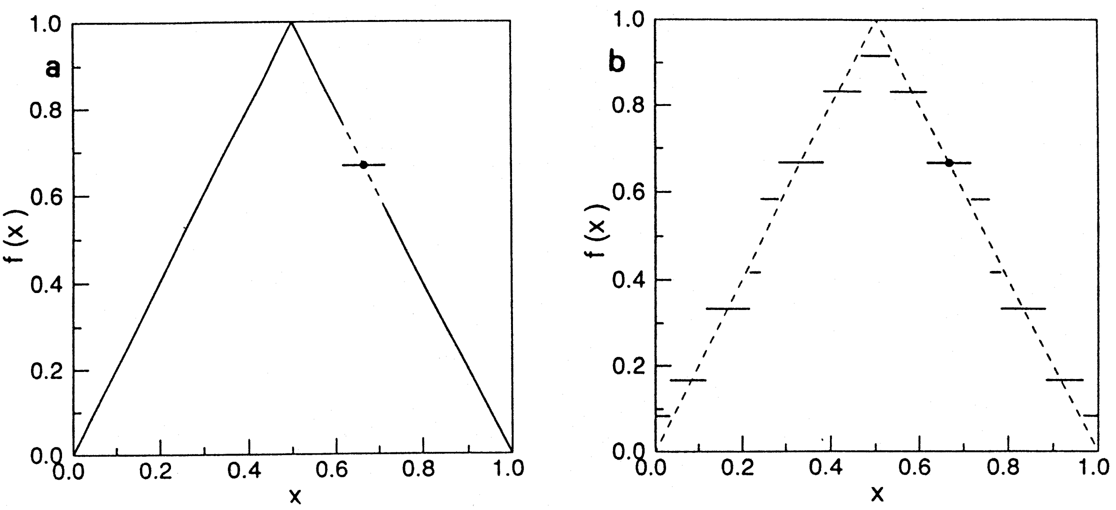

Fig. 2. Controlling tent map: (a) One-step method with $u_*=0.1$; (b) multistep method with $u_*=0.1$. Transition to global control occurs at $N_T=5$.

We applied this method to the Hénon map

$$x_1(t+1)=1.4-x_1^2(t)+0.3x_2(t)+u\ ,$$
$$x_2(t+1)=x_1(t)\ . \qquad (10)$$

Initial conditions were chosen randomly on the attractor, and then both the OGY method and the multistep method described above were applied. The results for the calculated average number of iterations to achieve control $\langle\tau\rangle$ are presented in fig. 1. One can see that the multistep method significantly reduces $\langle\tau\rangle$.

We now discuss a transition in the behavior of the controlled system that occurs for sufficiently large u_* and N_{max}. For simplicity, let us consider a one-dimensional mapping with chaotic behavior (the tent map),

$$x_1(t+1)=f(x_1(t),u)=1+u-2|x_1(t)-0.5|\ ,$$
$$0\leqslant x_1\leqslant 1\ . \qquad (11)$$

Stabilization of the unstable fixed point $x_F=\frac{2}{3}$ with the one-step method gives the perturbed (purely dynamical!) map shown in fig. 2a. The fixed point is now stable, but it coexists with transient chaos. Applying the multistep control method as described above, we get a map perturbed also near the points $f^{-1}(x_F, 0), f^{-2}(x_F, 0), \ldots$, so the area of transient chaotic behavior decreases. For some N_T the last unstable periodic orbit of the tent map (here the fixed point at origin) disappears and the mapping looks like fig. 2b. There are no exponentially distributed transients now: for all initial conditions the controlled behavior (the fixed point x_F) establishes after a finite time $t\leqslant N_T$. This transition may be described as transition from local (in a vicinity of an unstable periodic orbit) to global (overall strange attractor) control, when not only an unstable periodic orbit becomes stable, but also all chaotic transients disappear.

In conclusion we would like to mention that the method proposed is close to the recent method of targeting trajectories of chaotic attractors [7]. In targeting, a trajectory is directed to a given region in the phase space using small perturbations of the system. Our multistep control may be considered as a simple version of targeting, where only linear approximation near an unperturbed trajectory is used. In fact, we try to direct the trajectory in the vicinity of the fixed point using mainly maximal values of the control parameter $\pm u_*$. This procedure is rather crude and may be further improved, but still gives significant gain comparing with the OGY control method.

We thank U. Dressler and A. Loskutov for useful discussions. A.P. acknowledges the support from the Alexander von Humboldt Stiftung and from the Max Planck Gesellschaft.

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