The simplest case of chaotic wave scattering

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We consider scattering of nondispersive linear waves on a discrete nonlinear element. The problem reduces to the dynamics of a forced damped nonlinear oscillator. Chaotic motions of the oscillator produce chaotic reflected and transmitted waves.

The problem of chaotic scattering has received sufficient interest recently.[1,2] In classical mechanics the problem is of particle motion in unbounded two- or three-dimensional (or time-dependent) potentials. Quantum chaotic scattering has been also widely discussed.[3,2] In the latter case one has to analyze the Schrödinger equation—a linear partial differential equation. Completely similar is the problem of electromagnetic wave scattering on the objects with complicated geometry.[4] In all problems mentioned above the wave equations are linear and chaos appears as irregular ray trajectories.

In this paper we consider *nonlinear* wave scattering. The wave field is still supposed to be linear, but the scattering object is nonlinear. In fact, problems of this type have been already considered in plasma physics (chaotic stimulated Brillouin scattering[5]) and in nonlinear optics (Ikeda's system[6]). Here we present the simplest example of chaotic nonlinear wave scattering, which is easily solved by reducing the system to the equations of a forced damped nonlinear oscillator.

Consider transverse oscillations of an elastic string, coupled with a localized nonlinear oscillator (Fig. 1). Transverse displacements of the string $y(x,t)$ obey the linear wave equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0, \quad c^2 = \frac{T}{\rho}. \tag{1}$$

Here ρ and T are string's density and tension, respectively. A nonlinear oscillator $z(t)$ is coupled to the string at $x=0$ and is governed by the equation

$$m \frac{d^2 z}{dt^2} + Q(z) = T\left(\frac{\partial y^+(0,t)}{\partial x} - \frac{\partial y^-(0,t)}{\partial x}\right), \tag{2}$$

where $Q(z)$ is a nonlinear function. Equations (1) and (2) should be complemented with the boundary condition

$$z(t) = y^-(0,t) = y^+(0,t), \tag{3}$$

where y^- and y^+ are displacements of the string for $x<0$ and $x>0$, respectively. Assume that there is an incident wave coming from $-\infty$:

$$y_{\text{inc}}(t,x) = A \cos\left(\omega t - \frac{\omega}{c} x\right).$$

The problem is to find reflected and transmitted waves.

For the wave equation (1) the displacement y can be represented in the form

$$y^{\pm} = g^{\pm}\left(t - \frac{x}{c}\right) + f^{\pm}\left(t + \frac{x}{c}\right).$$

The causality principle requires that $g^- = y_{\text{inc}}$ and $f^+ = 0$. Then from the boundary condition (3) we find

$$\begin{aligned} f^-(t) &= z(t) - g^-(t) = z(t) - y_{\text{inc}}(t,0), \\ g^+(t) &= z(t). \end{aligned} \tag{4}$$

Substituting in (2) we finally obtain

$$m \frac{d^2 z}{dt^2} + \frac{2T}{c}\frac{dz}{dt} + Q(z) = -\frac{2T}{c} A\omega \sin(\omega t). \tag{5}$$

Equation (5) is an ordinary differential equation describing a periodically forced damped nonlinear oscillator. Many systems of this type have been shown to have chaotic solutions (see, e.g., Refs. 7 and 8). Substituting the solution of Eq. (5) in (4) we immediately get waveforms of reflected f^- and transmitted g^+ waves. An example of the power spectrum of the reflected wave for the Toda-type nonlinearity in Eq. (2) is presented in Fig. 2.

We conclude with the following remarks.

First, it is worth noting that although there is no dissipation in the original system (1) and (2), the resulting equation (5) is dissipative. This is of course radiation dissipation.

Second, the incoming wave is the only driving force in the system, and on its form depends the observed behavior. For example, if Eq. (5) has several attractors, then the regime of scattering depends on the way the incident wave is switched on. If the parameters of the incident wave are slowly modulated, then transitions between different attractors may be observed. If Eq. (5) demonstrates only

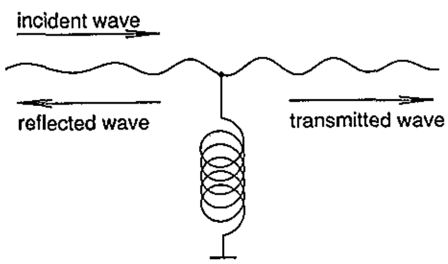

FIG. 1. Geometry of wave scattering.

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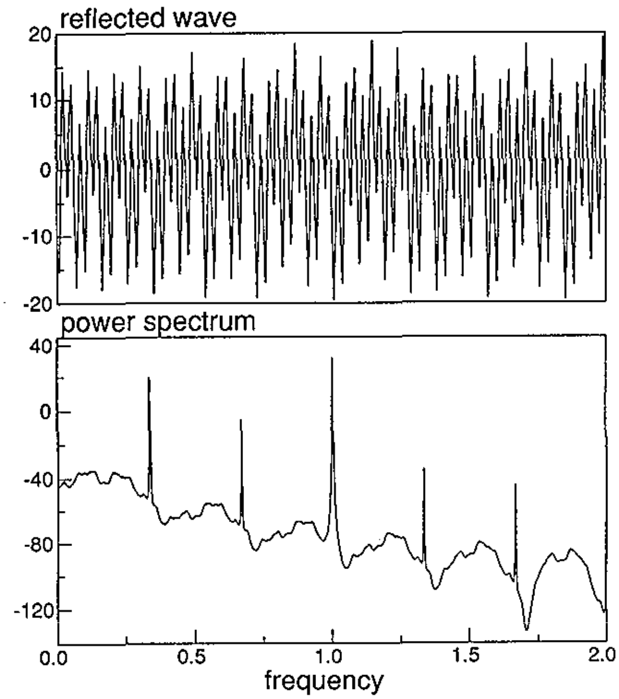

FIG. 2. Reflected wave and its spectrum for the Toda oscillator $\ddot{x}+0.2\dot{x}+e^{x}-1=-4\sin(1.6t)$. The power spectrum (obtained from the computed time series via averaging of the absolute values of Fourier transforms, as described in Ref. 9) is presented in the logarithmic scale (dB) versus frequency measured in units of the incident wave frequency ω. Sharp peaks should in fact be delta functions corresponding to the periodic component of the reflected wave; peaks at the frequencies $n\omega/3$ correspond to the subharmonic resonance (on the Poincaré map the attractor consists of three pieces).

transient chaos, then for relatively short incident pulses scattered waves will be chaotic, while for longer pulses after transient chaotic behavior regular scattering will be observed.

Third, we were able to obtain a nice closed ordinary differential equation (5) only because we considered nondispersive waves. If the waves are dispersive (and governed, e.g., by the Klein–Gordon equation) the problem of chaotic wave scattering reduces to an integrodifferential equation. The problem also becomes more complicated if there are several scatters—then instead of (5) we get a system of coupled nonlinear differential-delay equations.

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