Local Lyapunov exponents for spatiotemporal chaos

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Local Lyapunov exponents are proposed for characterization of perturbations in distributed dynamical systems with chaotic behavior. Their relation to usual and velocity–dependent exponents is discussed. Local Lyapunov exponents are analytically calculated for coupled map lattices using random field approximation. Boundary Lyapunov exponents describing reflection of perturbations at boundaries are also introduced and calculated.

I. INTRODUCTION

Chaotic motion in nonlinear dynamical systems is characterized by "sensitive dependence on initial conditions." Quantitatively, this sensitivity is measured by the Lyapunov exponent.[1] Chaotic motion in distributed dynamical systems with many degrees of freedom is often called spatiotemporal chaos. Straightforward application of Lyapunov exponent concept to spatiotemporal chaos leads to notion of the Lyapunov exponent density.[2–4] However, in distributed systems stability properties are much more complicated than in nondistributed ones, and, e.g., it is possible to define separately temporal and spatial Lyapunov exponents.[5,6]

Already linear stability properties of a ground state in a distributed system are nontrivial. Dispersion relations allows one to distinguish between stable and unstable situations, while a more thorough analysis is needed to distinguish between two possible cases of instability—absolute and convective.[7] In the case of absolute instability an initial spatially localized perturbation grows at the same place where it was imposed. In the case of convective instability initially localized perturbation moves away as it grows. The difference between these two types of instabilities is important because they demonstrate different types of turbulent behavior. In the systems with absolute instability there is sensitive dependence on initial conditions and spatiotemporal chaos is similar to chaotic motion in nondistributed systems. Well–known examples in hydrodynamics are Rayleigh–Bénard convection and Taylor–Couette flow.[8] In the case of convective instability there is no sensitive dependence on initial conditions (if one considers a system of finite length), but the system is sensitive to external perturbations. So the problem is how external perturbations (noisy[9] or regular[5,10]) are transformed into spatiotemporal turbulence. As examples of convectively unstable systems we can mention boundary layer, wind waves, and also Rayleigh–Bénard convection and Taylor–Couette vortices with imposed throughflow.[11–15]

In order to distinguish between absolute and convective instability it is not sufficient to consider linear perturbations of the ground state, but also secondary perturbations of the turbulent state. Then, if both are of convective type, the system may be called a "flow system." In Ref. 16 an example was given of a system with convective linear but absolute secondary instability.

One tool suitable to study secondary instabilities is the velocity–dependent Lyapunov exponent.[17] This quantity shows how perturbations grow in a reference frame moving with a constant velocity. Calculation of this exponent is however not simple.

In this paper we propose to characterize secondary instabilities with local Lyapunov exponents. They measure how a perturbation localized in space grows and spreads. For statistically spatially homogeneous systems they coincide with velocity–dependent Lyapunov exponents, but are much more easy to compute. These quantities may be also useful in characterizing nonhomogeneous states, in particular behavior of perturbations near boundaries.

The paper is organized as follows. Definition and examples of numerical calculations of local Lyapunov exponents are given in Sec. II. In Sec. III local Lyapunov exponents are calculated analytically for coupled map lattices, using approach of Ref. 18. In Sec. IV we define and calculate the boundary Lyapunov exponent. Section V contains concluding remarks.

II. LOCAL LYAPUNOV EXPONENTS

Spatiotemporal chaos may be observed in models of different types: partial differential equations, finite–difference equations, coupled map lattices. In this paper we shall consider only the simplest case—coupled map lattices (CML),[19] possible generalizations to continuous-time systems will be discussed elsewhere.

In a CML a field $u(x,t)$ depends on discrete space $x=1,2,3,\ldots$ and time $t=0,1,2,\ldots$ and satisfies an equation

$$u(x,t+1)=\hat{D}f(u(x,t)) \tag{1}$$

with appropriate boundary conditions. Here $f(\cdot)$ is a nonlinear function and $\hat{D}$ is a linear operator, defined on the discrete lattice. For nearest neighbors coupling one can write

$$\hat{D}u(x)=g_-u(x-1)+g_0u(x)+g_+u(x+1),$$
$$g_0+g_-+g_+=1. \tag{2}$$

The most widely used example is the symmetrical CML with

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$$g_- = g_+ = \frac{\varepsilon}{2}, \quad g_0 = 1 - \varepsilon. \tag{3}$$

Let us consider small perturbations of a turbulent state $u^0(x,t)$ in the system (1):

$$u(x,t) = u^0(x,t) + \mu w(x,t), \quad \mu \ll 1.$$

Linearizing Eq. (1) we obtain

$$w(x,t+1) = \hat{D} f'(u^0(x,t)) w(x,t). \tag{4}$$

Consider solution of Eq. (4) with *local* initial condition at a time t':

$$w(x,t') = \begin{cases} 1 & \text{if } x = x', \\ 0 & \text{else.} \end{cases} \tag{5}$$

We expect this solution to grow in time exponentially and define the local Lyapunov exponent as the averaged growth rate

$$\lambda(x,t;x',t') = \frac{1}{t-t'} \langle \ln | w(x,t;x',t') | \rangle. \tag{6}$$

If the turbulent state $u^0(x,t)$ is statistically homogeneous in space and time, we expect that the behavior of perturbations is similar to that in the linear homogeneous problem,[16] where the exponent is constant along the rays of constant velocity. [It should be emphasized that this is true only for statistically homogeneous states; if there are nonhomogeneities (e.g., boundaries, see IV below) the full form (6) should be used.] This means that the exponent (6) depends only on

$$v = \frac{x - x'}{t - t'} \tag{7}$$

thus giving

$$\lambda(v) = \lim_{t-t' \to \infty} \frac{1}{t-t'} \langle \ln | w(x' + v(t-t'), t; x', t') | \rangle. \tag{8}$$

The form (7)–(8) is natural, because we expect that the perturbation grows exponentially, propagates with some velocity and spreads due to diffusion. Thus

$$w \sim \exp(\alpha t) \exp\left(\beta \frac{(x - Vt)^2}{t}\right) \sim \exp\left(t\left[\alpha + \beta\left(\frac{x}{t} - V\right)^2\right]\right)$$

giving dependence of the exponent only on the x/t. For stability of ground steady state $u^0(x,t)$ = const the local Lyapunov exponent coincides with the convective exponent, defined in Ref. 16. Character of instability is clearly seen from the local Lyapunov exponent: if $\lambda(0) < 0$ while $\max_v \lambda(v) > 0$, secondary instability is of convective type; if $\lambda(0) > 0$, instability is absolute.

We now argue that the local Lyapunov exponent coincides with the velocity – dependent Lyapunov exponent, defined in Ref. 17. In the definition of velocity – dependent Lyapunov exponent an initial perturbation is localized in a region $x_1 < x \leqslant x_2$, and the disturbance is followed along the strip $x_1 + vt < x \leqslant x_2 + vt$ in order to obtain exponent corresponding to the velocity v. In Ref. 17 it was suggested to take $x_2 - x_1$ sufficiently large. This is, however, not necessary, because for large t both the initial interval $x_1 < x \leqslant x_2$ and the strip are much smaller than the overall spread of perturbation $x_1 + v_{min} t < x < x_2 + v_{max} t$, where v_{min} and v_{max} are minimal and maximal velocities of perturbations propagation [e.g., for the CML (1)–(3) $v_{max} = -v_{min} = 1$]. Thus, the size of initial perturbation does not play a role (see also discussion in Ref. 20) and we can set $x_1 = x_2$.

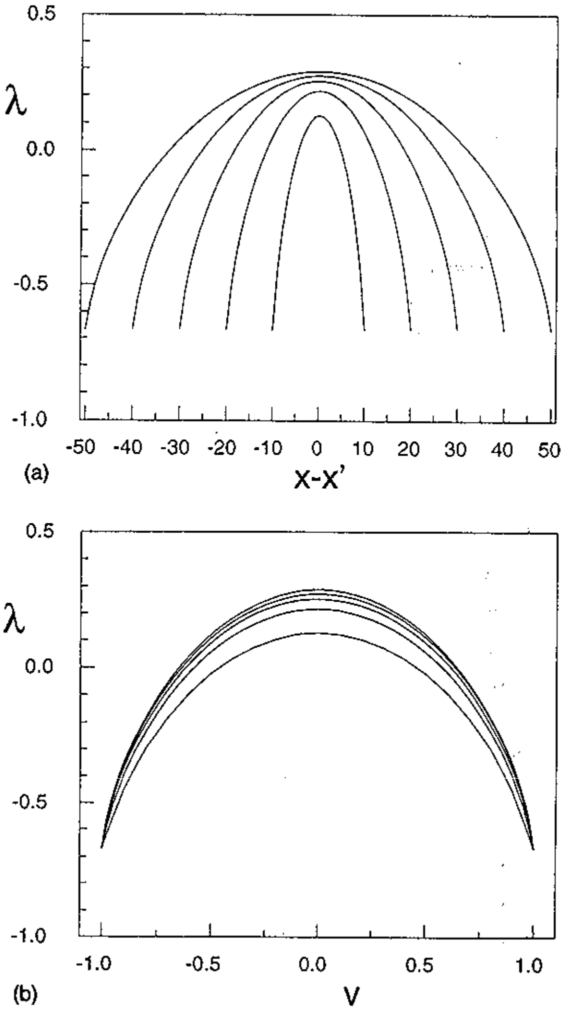

FIG. 1. Example of the calculation of the local Lyapunov exponent for the CML (1)–(3) with $\varepsilon = 2/3$, $f(u) = 4u(1-u)$. (a) Growth rate $\lambda(x,t;x',t')$ Eq. (6) for $t - t' = 10; 20; 30; 40; 50$. (b) The same exponents as functions of $v = (x - x')/(t - t')$.

One advantage of the local Lyapunov exponent is that it may be also defined for nonhomogeneous systems. For example, near boundaries they characterize reflection of perturbations (see Sec. IV below). Albeit in infinite statistically homogeneous systems the local Lyapunov exponents are equivalent to velocity–dependent ones, the former are more convenient to calculate. Instead of considering large systems, in calculating local Lyapunov exponents we may get large statistics by averaging over different x' and t'. Figure 1 shows that convergence with time is rather fast: it is sufficient to take $t_{max} = 50$ to obtain $\lambda(v)$ with good accuracy. Thus, we may consider systems of moderate size: $L \simeq t_{max} | v_{max} - v_{min} |$. This is an im-

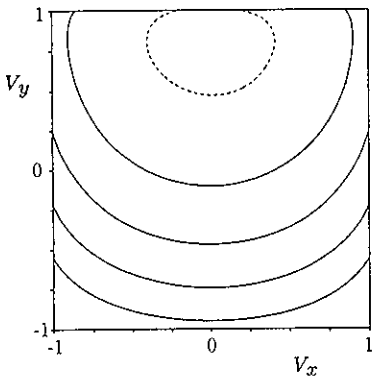

FIG. 2. The local Lyapunov exponent for a two-dimensional CML $v(x,y)=1-2|u(x,y)|$, $\bar{u}(x,y)=[(1+\gamma)/9][v(x-1,y-1)+v(x,y-1)+v(x+1,y-1)]+\frac{1}{9}[v(x-1,y)+v(x,y)+v(x+1,y)]+[(1-\gamma)/9][v(x-1,y+1)+v(x,y+1)+v(x+1,y+1)]$ with $\gamma=0.8$. The contours of the exponent as a function of v_x and v_y are presented for $\lambda=0$ (dashed line) and for $\lambda=-0.5; -1; -1.5; -2$. Because $\lambda(0,0)<0$, secondary instability is of convective type.

portant computational advantage allowing one to compute local Lyapunov exponents for two-dimensional systems (see Fig. 2).

Consider now the relation between the local Lyapunov exponents and the usual Lyapunov exponent. The latter depends not only on the local dynamics, but also on boundary conditions. A finite size system with open flow (nonreflecting) boundary conditions [in Ref. 21 the term "open boundary conditions" was misleadingly used for zero – derivative reflecting boundary conditions, see Sec. IV below] may be considered as a part of infinite size system, where perturbation is followed in the strip $0 < x < L$. This strip moves with zero velocity, so in this case the usual Lyapunov exponent is $\lambda(0)$. In the case of periodic boundary conditions the usual Lyapunov exponent is the maximal value of $\lambda(v)$.[20]

III. ANALYTIC ESTIMATION OF LOCAL LYAPUNOV EXPONENTS

In this section we calculate local Lyapunov exponents analytically for some simplest CML of type Eqs. (1)–(2). The method of Ref. 18 is used, which in turn follows the random energy model in the theory of directed polymers.[22,23] Similar approach was used in Refs. 3 and 24 for calculating Lyapunov exponents in symplectic mappings. In application to CML this method may be called "random field approximation," because spatiotemporal chaos is considered as uncorrelated random field.

Let us start with writing explicitly the equations for the perturbations in the CML (1),(2):

$$\begin{aligned} \tilde{w}(x,t) &= a(x,t)w(x,t), \\ w(x,t+1) &= g_-\tilde{w}(x-1,t)+g_0\tilde{w}(x,t)+g_+\tilde{w}(x+1,t). \end{aligned} \tag{9}$$

Here we denoted $a(x,t) = f'(u^0(x,t))$.

The main assumption underlying approach of Ref. 18 is to consider $a(x,t)$ as independent random variables (these quantities will be called hereafter instant growth rates). The only characteristics of $a(x,t)$ is then probability distribution density, mainly (but not totally) defined by the local dynamical equation $u(t+1) = f(u(t))$.

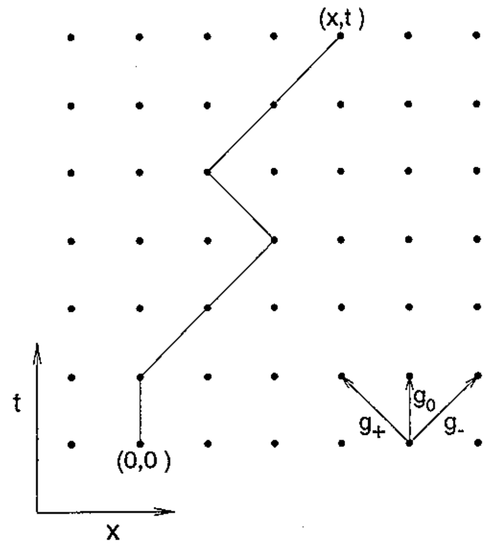

FIG. 3. Sketch of perturbation propagation on the plane (x,t). A path, connecting points (0,0) and (x,t) resembles a "directed polymer" (Ref. 23).

A. Constant instant growth rates

Let us start with the simplest possible case $a(x,t)=A=\text{const}$. This corresponds, e.g., to the mapping $f(u) = Au \pmod 1$ (usual Lyapunov exponents for this model were calculated in Ref. 4). Then the equation for the perturbations is the same for turbulent and ground states. Evolution of perturbation may be considered as a graph on the (x,t) plane (Fig. 3). Each point (x,t) contributes to the point $(x,t+1)$ with factor g_0 and to the points $(x+1,t+1)$ and $(x-1,t+1)$ with factors g_- and g_+. Also, at each node of the graph the perturbation is multiplied by A. Thus, to the field at a point (x,t) contribute all paths connecting this point with the origin (point of initial perturbation). Let us denote the number of all paths connecting (0,0) with (x,t) and having exactly m links with factor g_0 by $C(m,t,x)$. Because total number of links in these paths equals t, there are $N_- = (t-m+x)/2$ links with factor g_- and $N_+ = (t-m-x)/2$ links with factor g_+. Total number of such paths is

$$C(m,t,x)=\frac{t!}{m!N_-!N_+!} \tag{10}$$

and each of these paths contributes with the factor

$$G(m,t,x)=g_0^m g_-^{N_-} g_+^{N_+}.$$

For large t, using Stirling's formula and denoting

$$y=\frac{m}{t}, \quad v=\frac{x}{t},$$

we get

$$C(m,t,x)\approx\exp[tc(y,v)],$$
$$G(m,t,x)\approx\exp[tg(y,v)],$$

where

$$c(y,v)=-y\ln y-\tfrac{1}{2}(1-y+v)\ln\tfrac{1}{2}(1-y+v)$$
$$-\tfrac{1}{2}(1-y-v)\ln\tfrac{1}{2}(1-y-v),$$

$$g(y,v)=y\ln g_0+\tfrac{1}{2}(1-y+v)\ln g_-$$
$$+\tfrac{1}{2}(1-y-v)\ln g_+.$$

Now we sum all contributions with different m and take into account factor A^t:

$$w(x,t)\approx\int dy\exp[t(c(y,v)+g(y,v)+\ln A)].$$

Estimating the integral for large t using saddle–point method, we get

$$\lambda(v)=\ln A+c(y_0,v)+g(y_0,v),$$

where y_0 is obtained from the equation

$$c_y'(y_0,v)+g_y'(y_0,v)=0$$

giving

$$y_0=(1-v^2)(1+[v^2+(1-v^2)(4g_-g_+g_0^{-2})]^{1/2})^{-1}. \quad (11)$$

It is easy to verify that in the symmetrical case (3) we get $\lambda(0)=\ln A$.[18] We conclude this section noting that because all $a(x,t)$ are constant, no approximation was made and the obtained expressions for local Lyapunov exponent are exact.

B. Instant growth rates with random signs

Consider now the case when instant growth rates can change signs, but their absolute values are constant: $a(x,t)=\pm A$. This corresponds, e.g., to the tent local map

$$u(t+1)=f_{tent}(u(t))=1-A|u(t)|. \quad (12)$$

Again we present $w(x,t)$ as a sum of contributions from different paths, but now these contributions may have different signs. To take this into account we use approach of Ref. 23. Let us assume that

$$a(x,t)=\begin{cases}A & \text{with probability } \mu,\\ -A & \text{with probability } 1-\mu.\end{cases}$$

Then

$$\prod_{i=1}^{t}a(\xi_i,\tau_i)=\begin{cases}A^t & \text{with probability } P(t)\\ -A^t & \text{with probability } 1-P(t),\end{cases} \quad (13)$$

where

$$P(t)=\tfrac{1}{2}+\tfrac{1}{2}(2\mu-1)^t=\tfrac{1}{2}\pm\tfrac{1}{2}\exp(pt),\quad p=\ln|2\mu-1|.$$

So $w(x,t)$ is a sum of large number of terms whose sign is random with probability given by (13). Applying central limit theorem, we can suppose that $w(x,t)$ obeys Gaussian distribution with mean Q and variance B. To obtain the mean value we have to sum all terms with their signs:

$$Q=\sum C(m,t,x)G(m,t,x)(2P(t)-1)A^t$$

$$\approx\pm\int dy\exp(t[c(y,v)+g(y,v)+p+\ln A]).$$

This integral is dominated by the saddle point (11):

$$Q\approx\pm\exp(t[c(y_0,v)+g(y_0,v)+p+\ln A]).$$

Calculating the variance as a sum of variances of different terms we get

$$B=\sum C(m,t,x)G^2(m,t,x)(1-(2P(t)-1)^2)A^{2t}$$

$$\approx\int dy\exp(t[c(y,v)+2g(y,v)+2\ln A]). \quad (14)$$

In order to calculate the integral (14) we have to maximize the exponent:

$$c_y'(y_1,v)+2g_y'(y_1,v)=0$$

that gives

$$y_1=(1-v^2)(1+[v^2+(1-v^2)4(4g_-g_+g_0^{-2})^2]^{1/2})^{-1}, \quad (15)$$

so that

$$B\approx\exp(t[c(y_1,v)+2g(y_1,v)+2\ln A]).$$

Having calculated mean Q and variance of B of the Gaussian distribution for w, we can now express the Lyapunov exponent as

$$\lambda(v)=t^{-1}\langle\ln|w(v,t)|\rangle$$

$$=t^{-1}\int dw(2\pi B)^{-1/2}\ln|w|\exp\left(-\frac{(w-Q)^2}{2B}\right).$$

Because both Q and B grow exponentially with time, either $B\ll Q^2$ or $B\gg Q^2$, so

$$t\lambda(v)=\begin{cases}\ln|Q| & \text{if } Q^2\gg B\\ \tfrac{1}{2}\ln B & \text{if } Q^2\ll B.\end{cases}$$

Thus we have two phases:

(1) Mean dominated phase

$$\lambda(v)=t^{-1}\ln|Q|=\ln A+p+c(y_0,v)+g(y_0,v),$$

(2) Variance dominated phase

$$\lambda(v)=t^{-1}\tfrac{1}{2}\ln B=\ln A+\tfrac{1}{2}c(y_1,v)+g(y_1,v) \quad (16)$$

and the point of the "phase transition" is

$$p+c(y_0,v)+g(y_0,v)=\tfrac{1}{2}c(y_1,v)+g(y_1,v). \quad (17)$$

Formulas (16)–(17) are rather difficult for complete analyses, so we consider only the symmetric case (3) and the usual Lyapunov exponent $\lambda(0)$. For $v=0$ we have

$$c(y_0,0)+g(y_0,0)=0,$$

$$\tfrac{1}{2}c(y_1,0)+g(y_1,0)=\tfrac{1}{2}\ln(g_0^2+2g_-g_+)$$
$$=\tfrac{1}{2}\ln[(1-\varepsilon)^2+\varepsilon^2/2]$$

so the Lyapunov exponent is

$$\lambda(0)=\ln A+\hat{\lambda}(\varepsilon,p), \quad (18)$$

where

$$\hat{\lambda}(\varepsilon,p)=\max(\tfrac{1}{2}\ln[(1-\varepsilon)^2+\varepsilon^2/2],p). \quad (19)$$

If the probabilities for $a(x,t)$ to have plus and minus signs are equal, then $\mu=1/2$ and $p=-\infty$, so there is no phase transition. Transition may occur only when μ is close to 0 (or 1).

We checked formulas (18) and (19) for coupled tent maps (13) with $A=2$ (see Fig. 4). Indeed, λ vs ε behavior resembles phase transition, but the value of μ that fits data $\mu_{fit}=0.11$ is very far from the empirical probability, obtained directly from CML: $\mu_{emp}=0.29$. This discrepancy might be caused by high correlations between $a(x,t)$ as seen from Fig. 4. To check this we calculated the Lyapunov exponent for CML with doubled tent map $u(t+1)=f_{tent}^2(u(t))$. This CML gives much less correlations between $a(x,t)$ (still these quantities have constant absolute values) and the resulting Lyapunov exponent fits the theoretical prediction (20) rather well.

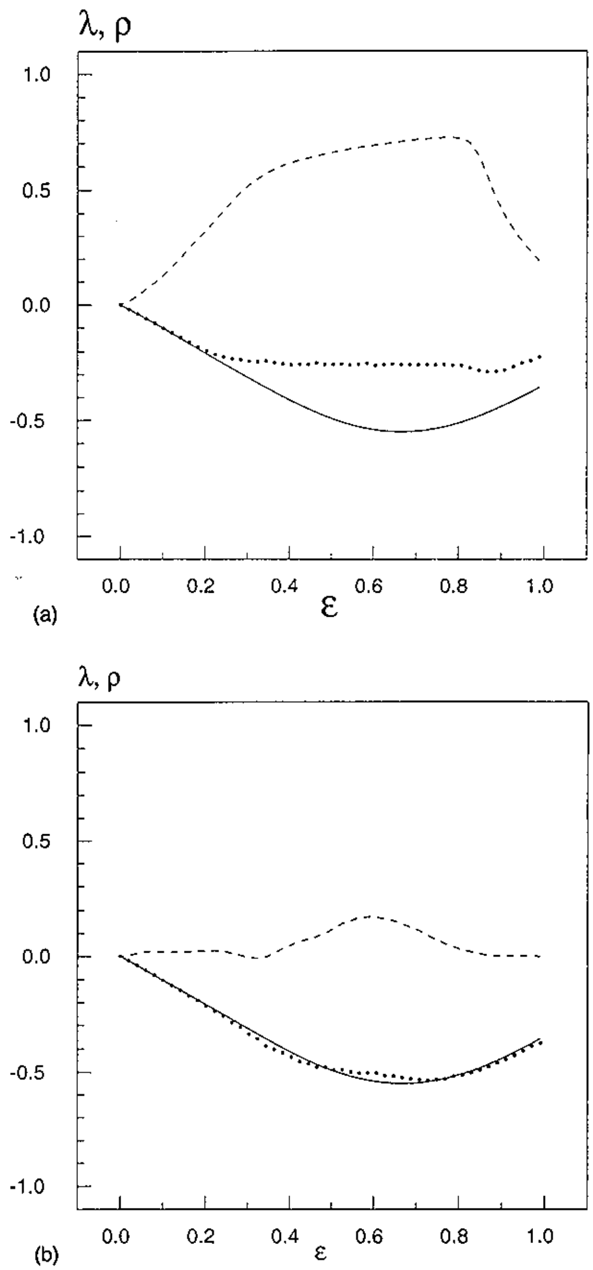

FIG. 4. The usual Lyapunov exponent $\tilde{\lambda}$ (dots) and the nearest-neighbor correlation of the field $a(x,t)$ ρ (dashed line). The solid line—theoretical formula (19). (a) The CML with the tent map (12). (b) The CML with the doubled tent map.

C. General case

Let us now consider a more general case, when instant growth rates fluctuate. For simplicity of presentation we shall suppose that these growth rates have signs plus and minus with probability 1/2. This eliminates the possibility of transition between mean-dominated and variance-dominated phases, as described above.

Again, as it was done previously, we must sum contributions from all paths connecting points (0,0) and (x,t), but now the product of instant growth rates gives not A^t, but

$$\prod_{i=1}^{t}|a(\xi_i,\tau_i)|.$$

Because we assume that the quantities $a(\xi_i,\tau_i)$ are independent random variables,

$$\prod_{i=1}^{t}|a(\xi_i,\tau_i)|=\exp\Lambda t$$
$$\text{with probability } R(\Lambda,t)\sim\exp[t\,r(\Lambda)],$$

where $r(\Lambda)$ is obtained from the central limit theorem

$$r(\Lambda)=-\frac{(\Lambda-\Lambda_0)^2}{2M},$$
$$\text{where } \Lambda_0=\langle\ln|a|\rangle,\ M=\langle(\ln|a|-\Lambda_0)^2\rangle.$$

In fact, one can consider a more general form of scaling function $r(\Lambda)$ appearing from chaotic dynamics,[25] but it is not clear if going only in this point beyond the approximation of independent instant growth rates gives any improvement.

Now, because at the point (x,t) we sum terms with different signs, for every y and Λ each term has absolute value $\exp[t(g(y,v)+\Lambda)]$ and the number of such terms is $\exp[t(c(y,v)+r(\Lambda))]$. Summing these terms we get Gaussian distribution with the variance

$$\tilde{B}\approx\int dy\,d\Lambda\,\exp[t(2g(y,v)+2\Lambda+c(y,v)+r(\Lambda))].$$

Estimation of this integral at the maximum of exponent gives

$$B \approx \exp[t(2g(y_1,v)+2\Lambda_1+c(y_1,v)+r'(\Lambda_1))],$$

where y_1 is given by (15) and $r'(\Lambda_1) = -2$. Thus we get

$$\lambda(v)=\frac{\ln B}{2t}=g(y_1,v)+\Lambda_1+\frac{1}{2}[c(y_1,v)+r(\Lambda_1)].$$

This formula is valid if we indeed have a sum of a large number of terms, contributing to $w(x,t)$ for $y=y_1$, $\Lambda=\Lambda_1$. However, it may happen that

$$c(y_1,v)+r(\Lambda_1)<0$$

which means that we have an exponentially small number of these terms. In this case according to Ref. 18 the sum is dominated by one leading term: $c(y,v)+r(\Lambda)=0$ and

$$w(x,t)\approx \pm \int dy\, d\Lambda \exp[t(g(y,\Lambda)+\Lambda)].$$

Now the integral is estimated as

$$w(x,t)\approx \pm \exp[t(g(y_2,v)+\Lambda_2)],$$

where y_2, Λ_2 are solutions of the system

$$c(y_2,v)+r(\Lambda_2)=0,$$

$$g_y'(y_2,v)r'(\Lambda_2)-c_y'(y_2,v)=0.$$

We have two phases:

(1) "Gaussian" phase

$$\lambda(v)=g(y_1,v)+\Lambda_1+\tfrac{1}{2}(c(y_1,v)+r(\Lambda_1)),$$

(2) "Single–term" phase

$$\lambda(v)=g(y_2,v)+\Lambda_2.$$

One can easily see that $\lambda(v)$ changes continuously at the point of transition determined from the condition

$$c(y_1,v)+r(\Lambda_1)=0.$$

For symmetrical CML (3) the "single–term" phase occurs only for $|v| \approx 1$ and for $\varepsilon \ll 1$, $|v| \ll 1$. This phase, however, is responsible for the scaling of the usual Lyapunov exponent for small ε.[18]

IV. BOUNDARY LYAPUNOV EXPONENT

In the study of spatiotemporal chaos in finite length systems it is not sufficient to follow perturbations in the bulk, their behavior at the boundaries is also of great importance. In this section we introduce the boundary Lyapunov exponent to characterize boundary effects quantitatively.

Let us suppose that initial perturbation (5) is imposed near the boundary $x=0$ in a semi-infinite system $0\leqslant x<\infty$. Then Eq. (8) is valid only for $v\geqslant 0$. Now $\lambda(0)$ determines the behavior of perturbations at the boundary and we shall call it the boundary Lyapunov exponent λ_B. The perturbation may as well be imposed at the boundary itself, so

$$\lambda_B=\lim_{t-t'\to\infty}\frac{1}{t-t'}\langle \ln|w(0,t;0,t')|\rangle.$$

We shall calculate λ_B for the CML (1),(2) with the linear boundary condition

$$f(u(-1,t))=\alpha f(u(0,t)).$$

Note, that the boundary condition is defined not for a field u, but for a field on which the diffusion operator $\hat{D}$ acts. For the perturbation this boundary condition has a form

$$\tilde{w}(-1,t)=\alpha\tilde{w}(0,t).$$

We shall find the boundary Lyapunov exponent using the method of Ref. 20. Consider first the case of constant growth rate $a(x,t) = A =$ const. Let us suppose that the field near the boundary $x = 0$ has a form

$$w(x,t)\simeq \bar{w}\exp(\lambda_B t-\kappa x). \tag{20}$$

Substituting (20) into (9) and (10) for $x = 0$ and $x = 1$ we obtain

$$\exp(\lambda_B)=A(g_-\alpha+g_0+g_+\exp(-\kappa))$$

$$=A(g_-\exp(\kappa)+g_0+g_+\exp(-\kappa))$$

so that

$$\lambda_B=\ln(g_0+\alpha g_-+\alpha^{-1}g_+)+\ln A,$$

$$\kappa=\ln\alpha.$$

Let us discuss the dependence of λ_B on the parameter α. For $\alpha = 1$ (free boundary) the boundary Lyapunov exponent is equal to the maximal value of local Lyapunov exponent:

$$\lambda_B(\alpha=1)=\ln A=\max_v\lambda(v).$$

The minimal value of boundary Lyapunov exponent is equal to the local Lyapunov exponent at $v = 0$:

$$\min_\alpha\lambda_B=\ln(g_0+2(g_-g_+)^{1/2})+\ln A=\lambda(0),$$

$$\alpha_{\min}=g_+^{1/2}g_-^{-1/2}. \tag{21}$$

This boundary condition may be called nonreflecting, because in this case local pertubation near the boundary evolves exactly like in the infinite medium (see Fig. 5).

Behavior of perturbations near the boundaries is illustrated in Fig. 5. Figure 5(a) shows how initial perturbation is "reflected" at the boundaries in the system with free boundary conditions $\alpha = 1$. Eventually this perturbation evolves into an eigenvector of the largest usual Lyapunov exponent (here this eigenvector is a constant). Figure 5(b) shows that in the system with nonreflecting boundary conditions (21) the perturbation evolves almost exactly like in the infinite system.

Consider now the case when instant growth rates have signs plus and minus with equal probabilities (generalization to the case of nonequal probabilities is straightforward): $a(x,t) = \pm A$. We shall assume, as in Sec. III B that $w(x,t)$ are independent random variables with Gaussian distribution. Then the variance for $x = 0$ and $x = 1$ evolves as follows:

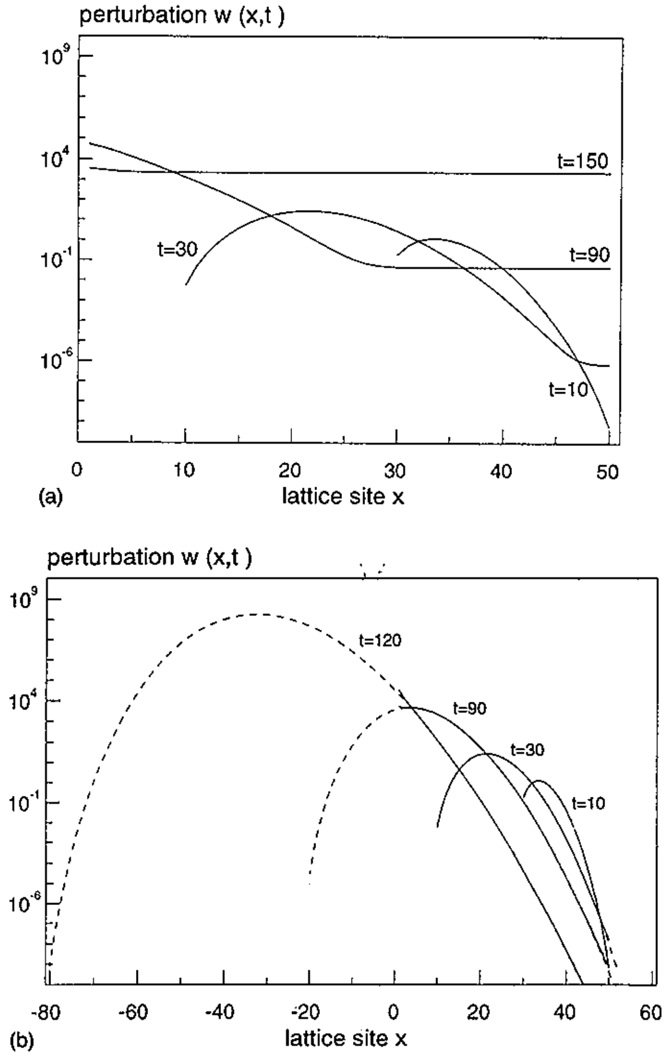

FIG. 5. Evolution of initial perturbation imposed at $x'=40$ in the CML (9) and (3) with $a(x,t)=1.2$, $g_+=0.7$, $g_0=0.2$; $g_-=0.1$; $1\leqslant x\leqslant 50$. (a) Free boundary conditions $\tilde{w}(0)=\tilde{w}(1)$; $\tilde{w}(51)=\tilde{w}(50)$. (b) Solid line—CML with nonreflecting boundary conditions $\tilde{w}(0)=(g_+/g_-)^{1/2}\tilde{w}(1)$; $\tilde{w}(51)=(g_-/g_+)^{1/2}\tilde{w}(50)$. Dashed line—the same perturbation in an "infinite" domain. In the region $1\leqslant x\leqslant 50$ the curves nearly coincide.

$$\langle w^2(0,t+1)\rangle = A^2(g_0+\alpha g_-)^2\langle w^2(0,t)\rangle + A^2 g_+^2\langle w^2(1,t)\rangle, \qquad (22)$$

$$\langle w^2(1,t+1)\rangle = A^2(g_-^2\langle w^2(0,t)\rangle + g_0^2\langle w^2(1,t)\rangle + 2g_+^2\langle w^2(2,t)\rangle). \qquad (23)$$

Assuming that

$$\langle w^2(x,t)\rangle \sim \exp(2\lambda_B t - 2\kappa x),$$

we get

$$\lambda_B = \tfrac{1}{2}\ln[g_0^2 + 2g_0 g_\alpha + g_-^2\alpha^2 + g_-^2 g_+^2(2g_0 g_\alpha + g_-^2\alpha^2)^{-1}] + \ln A,$$

$$\kappa = \tfrac{1}{2}\ln(2\alpha g_0 g_-^{-1} - g_0^2 g_-^{-2}).$$

Again, the minimal value of λ_B is equal to $\lambda(0)$ [see Eq. (19)]

$$\lambda_B(\alpha=\alpha_{\min}) = \tfrac{1}{2}\ln(g_0^2 + 2g_- g_+) + \ln A = \lambda(0)$$

and this boundary condition with

$$\alpha_{\min} = \pm(g_0^2 + g_- g_+)g_-^{-1} - g_0 g_-^{-1}$$

may be called nonreflecting.

V. CONCLUDING REMARKS

In this paper we have introduced the local Lyapunov exponent as the growth rate of a localized initial perturbation in a distributed system with chaotic behavior. This definition has sense if dynamics is local in space. As the simplest example we used a coupled map lattice with nearest–neighbor coupling. It seems that generalization to other systems which are continuous in space and/or in time is straightforward (while requiring much more computational efforts). In the systems with global coupling[26] the local exponent cannot be defined.

In the theory of chaos stability the effective Lyapunov exponents play an important role.[27,25] Effective, or instant exponents measure fluctuations of the growth rate of perturbations. This concept can be directly applied to local exponents, and is now under investigation. Another question is how local Lyapunov exponents are related to the spectrum of usual Lyapunov exponents, i.e., to the whole set of positive and negative exponents. In the linear case the local exponents can be expressed via dispersion relation for the Lyapunov spectrum.[16] For secondary perturbations the problem is more complicated (see Ref. 20).

In this paper we also applied the random field method to calculation of local Lyapunov exponents in CML analytically. In this method all correlations in spatiotemporal chaos are neglected. The method gives good results for highly chaotic systems like double tent map, while for the tent map corrections due to correlations are large. So it is not clear if the phase transitions, predicted by the random field model, may be observed in CMLs with such correlations.

Note added in proof. After this work was completed, I became aware of Ref. 28, where the method of Sec. II is applied to calculation of velocity– dependent Lyapunov exponents in the symmetric CML. The results of Sec. III A coincide with those of Ref. 28, while in the case of random signs (Sec. III B) a possibility of phase transition is not discussed in Ref. 28.

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