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SPATIAL DEVELOPMENT OF CHAOS

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ABSTRACT

Transition to turbulence in the systems with convective instability (flow systems) is investigated. It is shown that for deterministic external perturbations there is no dimension growth with distance. Transition to turbulence occurs through filling of the phase space by a low-dimensional object. Experimental observation of the transition in a simple electronic circuit is presented.

1. Introduction

Chaotic behavior of spatially extended dynamical systems is intensively investigated now. Such systems often show low-dimensional behavior, very similar to that in finite-order dynamical systems. For example, a field in a resonator may be considered as a finite set of discrete modes (higher modes damp out because of viscosity) and their evolution is governed by a strange attractor. The main property of chaos is sensitive dependence on initial conditions: small perturbations of initial field grow exponentially in time.

More complex situation occurs in infinite (or semi-infinite) in space media. Here already a linear stability problem is nontrivial. There are two kinds of instability in spatially extended systems: *absolute*, when perturbations grow at the same place where they were imposed, and *convective*, when perturbations move away as they grow. In the case of absolute instability sensitive dependence on initial field may occur and resulting turbulent behavior is similar to finite-dimensional chaos. For convective instability a localized initial disturbance simultaneously grows and moves away. So if we are interested in the field behavior in a finite spatial region, it is not sufficient to impose initial conditions. A nontrivial state may be observed only if there are constantly driving perturbations. In experiment these perturbations may be natural (usually random) or artificial (periodic, quasiperiodic, etc.). Thus, for convectively unstable systems we have the problem of nonlinear dynamical transformation of an external signal[1,2]. If the region of external driving is localized (e.g., at a boundary of a semi-infinite medium), then we may say about spatial amplification and development of perturbations.

As far as we are considering turbulent states in distributed systems, we must generalize notions of absolute and convective instability for nonlinear regimes. This may be done using velocity – dependent Lyapunov exponents[3]. If the Lyapunov exponent for zero velocity is positive, then the system is absolutely unstable to secondary perturbations. If the Lyapunov exponent for zero velocity is negative, while for some non–zero velocity there exist positive Lyapunov exponent, then the system is convectively unstable to secondary perturbations and we will call it *flow system.*

2. Examples of flow systems

2.1 Theoretical models

One of the widely used models for nonlinear nonequilibrium media is the generalized Ginzburg–Landau equation

$$\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} = a + (1+ic_1)\frac{\partial^2 a}{\partial x^2} + (-1+ic_2)|a|^2 a \tag{1}$$

Here linear velocity v is the parameter which determines either instability is convective or absolute. For $v = 0$ instability is absolute, while for large v it is convective. Note, that for Eq.1 with periodic boundary conditions there is no difference between absolute and convective instability and one can go from one case to another just changing a reference frame. For semi–infinite medium however one cannot change reference frame without changing boundary and initial conditions, so the v–term is important.

Another popular model is the Kuramoto – Sivashinsky equation

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0 \tag{2}$$

Here also the type of instability depends on the value of velocity v.

Much more simple then the partial differential equations (1),(2) are discrete coupled map lattices[4]. Flow systems may be modelled by lattices with unidirectional coupling, for example[5]

$$u_{n+1}(x) = (1-\varepsilon)f(u_n(x)) + \varepsilon f(u_n(x-1)) \tag{3}$$

Here n and x are discrete time and space. Because in linear derivatives matrix all elements above the diagonal are zero, Lyapunov exponents are easily expressed as

$$\lambda_x = ln(1-\varepsilon) + < ln|f'(u(x))| > \tag{4}$$

Thus, for ε close to 1 all Lyapunov exponents are negative and there is no absolute instability. If the mapping $u \to f(u)$ is chaotic, then for $\varepsilon = 1$ the Lyapunov exponent corresponding to the velocity 1 is positive, and the system is convectively unstable. By continuity, we conclude that it is convectively unstable also for ε close to 1.

Finally, I would like to describe a model, which is a purely flow system and cannot be transformed into absolutely unstable one. This is a set of unidirectionally coupled ordinary differential equations

$$\frac{du_x}{dt} + u_x = f(u_{x-1}) \qquad (5)$$

where $x = 1, 2, 3, \ldots$ is discrete "space". All Lyapunov exponents for eq.(5) are equal to -1, so there is stability to initial conditions. However, if the mapping $u \to f(u)$ is chaotic, then there is sensitive dependence to external field $u_0(t)$, which plays a role of boundary condition for this system.

2.2 Experimental systems

Hydrodynamic systems are the best example of situations, where convective instability occurs[6]. Transition to turbulence in such systems depends drastically on the nature of perturbations. In boundary layer experiments[7] it was shown, that applying periodic perturbations near the edge it is possible to generate regular spatially growing Tollmin – Schlichting waves. Their secondary instability results, however, in a very complicated flow structure. This transition is now very far from dynamical understanding. Much more simple appears to be more artificial situations, where mean flow is imposed over the fluid motion, known to exhibit low–dimensional chaos. Experiments on the Taylor – Couette flow with superimposed mean flow are now in progress[8]. Also, waves on the film, falling over vertical or inclined surface, in the first approximation are governed by the Kuramoto – Sivashinsky equation (2). However, I don't now experiments on the controlled transition to turbulence for this system.

In plasma a well–known example of convective instability gives plasma-beam system. Theoretically and numerically spatial development of chaos in this system was considered in ref[9].

In nonlinear optics flow systems correspond to unidirectional beam systems without feedback. Consider, for example, the Ikeda's system, which is essentially an interacting with laser beam nonlinear element, placed in a ring cavity. The system is described by difference–delay equation. Placing several nonlinear elements consequently and removing cavity, we obtain flow system, governed by a set of ODEs, similar to (5) (see[10], where such a set with periodic boundary conditions was considered).

In electronics, it is very easy to model Eq. (5) with a chain of nonlinear amplifiers[11]. Experiments with this chain will be presented below.

3. Does dimension grow in flow systems?

Recently, the attempts were made to describe quantitatively the spatial development of turbulence by measuring spatial growth of dimension[12]. Here I present simple arguments, showing that dimension does not grow in flow systems. It is known, that dimension of a dynamical system is limited from above by the Lyapunov dimension[13]. For flow systems, as they were defined above, all Lyapunov exponents are negative. However, we must take into account the Lyapunov exponents of the external signal, which is transformed by the flow system. It is natural to connect with a periodic signal one zero Lyapunov exponent, with a two-frequency quasiperiodic signal two zero exponents. For a chaotic signal it is natural to take into account as many Lyapunov exponents, as are nessesary for determination its Lyapunov dimension. Combining the Lyapunov exponents from the external signal and from the flow system, we can immediately conclude that there can be no chaotisation of periodic and quasiperiodic signals. The dimension of a chaotic external signal may grow a little, but this growth may be considered as a result of linear filtering of the input signal[14].

4. Spatial development of quasiperiodic signal

4.1 Theory

As was shown above, in flow systems periodic and quasiperiodic boundary signals remain to be periodic and quasiperiodic respectively. However evolution of quasiperiodic signal is nontrivial and in some sense may be described as spatial development of chaos.

Let us consider Eqs. (5) with $f(u) = 4u(1-u)$ being logistic map and quasiperiodic external signal $u_0(t) = U_0 + U_1\cos(\omega_1 t + \phi_1) + U_2\cos(\omega_2 t + \phi_2)$, the ratio ω_1/ω_2 being irrational. Then signal $u_1(t)$ has also harmonics of these frequencies $\pm\omega_1 \pm \omega_2$, $2\omega_1$, $2\omega_2$. These new frequencies grow with x and produce new combination frequencies $m\omega_1 + n\omega_2$, m and n being real numbers. All new frequencies grow and the spectrum becomes more and more dense (see fig.1). For large x this spectrum is almost undistinguishable from continuous one, but strictly speaking it is always discrete.

Another view on the spatial evolution of the field gives phase space. Two–frequency qusiperiodic motion corresponds to two–dimensional torus. Cross–section of this torus is topologically a circle, and this cross–section can be easy constructed by plotting $u^{(n+1)}$ versus $u^{(n)}$, where $u^{(n)} = u(n2\pi\omega_2^{-1})$. At fig.1 one can see how this topiological circle is stretched and folded with x. Its length grows exponentially, as shown on fig.2. For large x this "circle" fills the phase space densely, and is almost undistinguishable from a high–dimensional strange set.

4.2 Experiment

Experiments were performed with a chain of nonlinear amplifiers. Each

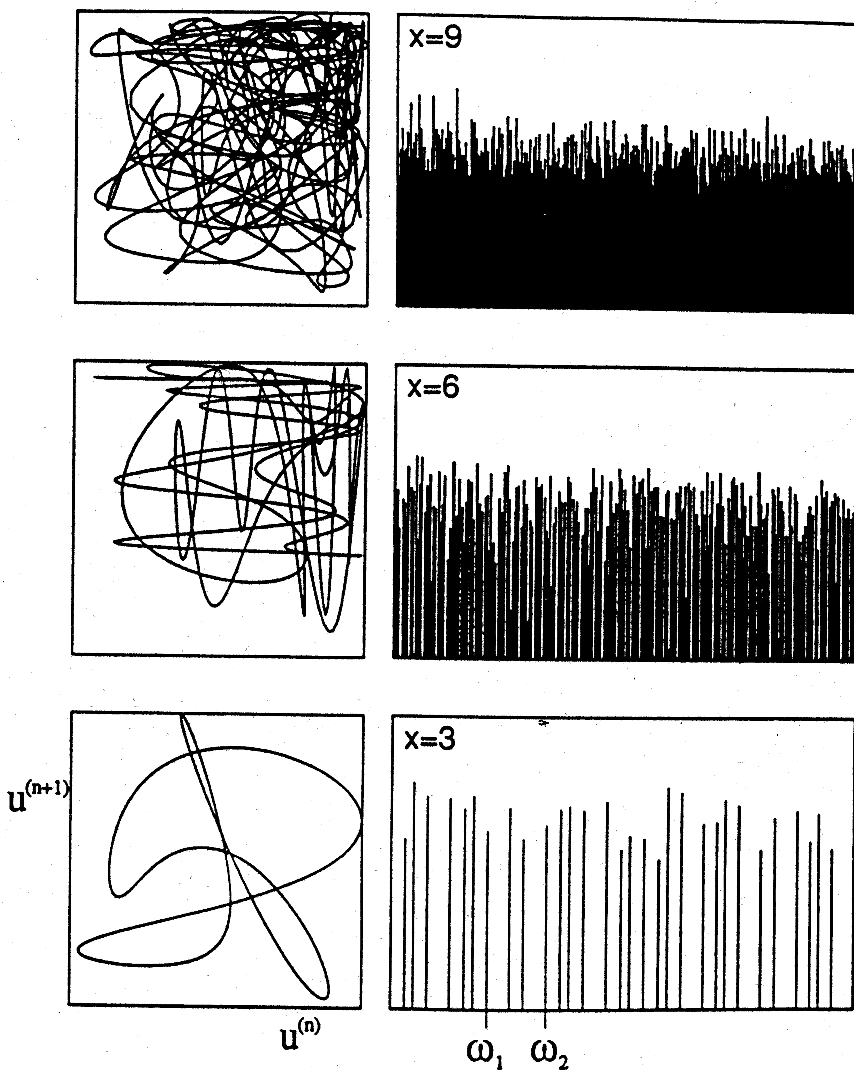

Figure 1. Evolution of the qusiperiodic signal with $\omega_2/\omega_1 = (\sqrt{5}+1)/2$ in the chain of amplifiers Eq. 5. Left: cross-sections of the two-torus. Right: spectrum (logarithmic scale).

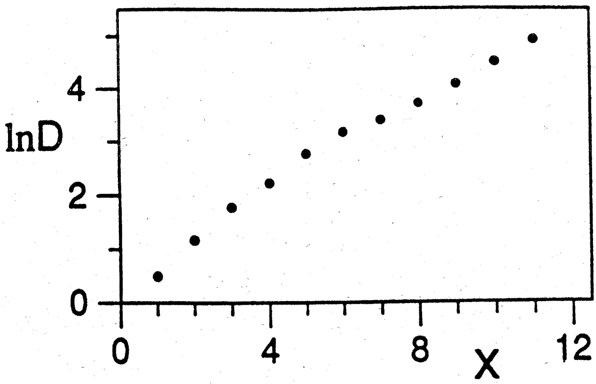

Figure 2. Evolution of the length D of the torus cross–section with distance x.

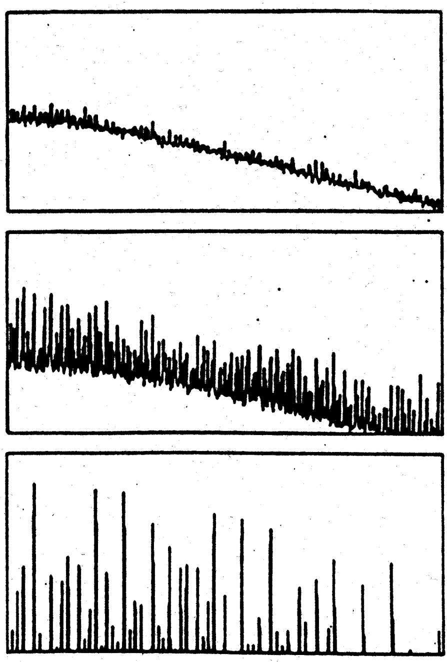

Figure 3. Experimentally obtained spectra (logarithmic scale) in the chain of nonlinear amplifiers. From bottom to top: $x = 10$, $x = 20$, $x = 30$.

element consisted of a nonlinear amplifier with $u_{out} = C_1 - C_2 u_{in}^2$ and a low-pass linear filter. The system is precisely described by the equations (5). The parameters C_1 and C_2 were chosen to provide the logistic map f near the point of fully developed chaos. Evolution of the spectrum of quasiperiodic input signal is shown on fig.3. The described above process of combination frequencies generation and amplification is clearly seen here. In addition, unavoidable noise also grows, and as a result at $x \simeq 30$ broad-band noise is observed.

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