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Discrete model of spatially mixing system

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We model a one-dimensional extended system which is subject to continuous spatial mixing with a mixing lattice transformation. Discrete and continuous in space mixing operators are juxtaposed. It is shown that the spatial field arising in the discrete system is close to self-affine Brownian motion.

Many attempts have been made during recent years to model turbulent behavior in spatially extended systems with coupled map lattices (CML) [1]. These lattices are discrete in space and time. Discreteness in time allows one to model easily both regular and chaotic dynamics, using appropriate nonlinear mappings. Discreteness in space provides a very simple computer coding, while allowing one to model such properties of distributed systems as diffusion and convective transport [2]. A wide class of CML may be represented in the form

$$u_{n+1}(x)=\hat{D}f(u_n(x)) . \qquad (1)$$

Here f is a nonlinear function which governs the dynamics of the point system (very often one uses the logistic mapping $f(u)=4u(1-u)$) and $\hat{D}$ is a linear operator, describing diffusion, convective transport, etc. The spatial coordinate x in (1) is, generally speaking, continuous, albeit in CML it is discrete.

Here we develop an analogous approach to modeling spatially distributed systems with mixing spatial dynamics. Such models arise naturally in chemical dynamics. Let us consider a chemical reaction in a solvent. The temporal dynamics of such a reaction may be rather complicated (chaotic) and described by a one-dimensional mapping [3]. Also, the whole solvent may be mixed in some regular way [4]. For example, in case of two spatial dimensions this mixing may be described by a mapping similar to the well-known baker transformation [5]. We restrict ourselves in this paper only to the case of one spatial dimension. Thus, we may introduce the following mixed map lattice (MML) model,

$$u_{n+1}(x)=\hat{M}f(u_n(x)) , \qquad (2)$$

where $u_n(x)$ could e.g. be a concentration of some chemical at time n. The main difference from (1) is in the nature of the mixing linear operator $\hat{M}$.

Consider first the case of one continuous spatial variable x, $0\leqslant x\leqslant 1$. We will assume that mixing is given by the simplest 2-adic transformation

$$x\mapsto 2x \quad (\mathrm{mod}\ 1) , \qquad (3)$$

then the operator $\hat{M}$ is nothing else but the Frobenius–Perron operator for the transformation (3) [6]:

$$\hat{M}u(x)=\tfrac{1}{2}[u(\tfrac{1}{2}x)+u(\tfrac{1}{2}(x+1))] . \qquad (4)$$

The easiest way to understand the properties of the mixing operator (4) is to consider the Fourier representation

$$u(x)=\sum_k a(k)\exp(2\pi \mathrm{i}kx) . \qquad (5)$$

Substituting (5) in (4) we obtain for the evolution under the linear operator $\hat{M}$

$$a_{n+1}(k)=a_n(2k) . \qquad (6)$$

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If the initial field is smooth, its Fourier spectrum has an exponential tail,

$$a_0(k) \sim \exp(-\alpha k) . \quad (7)$$

Then, from (6) and (7) we obtain for large n

$$a_n(k) \sim \exp(-2^n \alpha k) , \quad (8)$$

i.e. spectral harmonics decay with time faster than exponentially. This means that the mixing operator $\hat{M}$ smooths the field very efficiently and we may expect that in the full nonlinear system (2) a homogeneous solution will be established. Indeed, let us consider the stability of the homogeneous solution U_n. For a small nonhomogeneous disturbance $v_n(x)$ we get

$$v_{n+1}(x) = f'(U_n)\hat{M} v_n(x) . \quad (9)$$

Taking into account that

$$\prod_1^N |f'(U_n)| \propto \mu^N ,$$

where μ is the Lyapunov number of the mapping $u \mapsto f(u)$, we get for the Fourier components $b(k)$ of $v(x)$:

$$b_n(k) \sim \mu^n \exp(-2^n \alpha k) \to 0 \quad \text{for } k \neq 0 . \quad (10)$$

This conclusion is valid, however, only for smooth fields. In the case of a nonsmooth field $u(x)$ the Fourier spectrum has usually power asymptotics,

$$a_0(k) \sim k^{-\beta} , \quad (11)$$

and taking into account (6) we get only an exponential decay of the Fourier harmonics:

$$a_n(k) \sim (2^\beta)^n k^{-\beta} . \quad (12)$$

Considering now the stability of a homogeneous solution with respect to nonsmooth perturbations, we get a stability condition of the form

$$2^{-\beta}\mu > 1 \quad (13)$$

and instability may develop if μ is rather large.

The difference described above between smooth and nonsmooth fields shows that a discrete version of the model (2) may have nontrivial properties. Consider now the variable x as discrete: $x = 0, 1, 2, \ldots, N-1$. The straightforward analog of the Frobenius–Perron operator (4) has the form

$$\hat{M} = \tfrac{1}{2}(u[\tfrac{1}{2}x] + u[\tfrac{1}{2}(x+N)]) , \quad (14)$$

where $[y]$ is the integer part of y. Let us consider the properties of this operator using the discrete Fourier transform,

$$u(x) = \sum_{l=0}^{N-1} a(l) \exp[(2\pi/N)\mathrm{i}xl] ,$$

$$a(l) = \frac{1}{N} \sum_{x=0}^{N-1} u(x) \exp[-(2\pi/N)\mathrm{i}xl] ,$$

$$a(l+N) = a(l) . \quad (15)$$

Substituting (15) in (14), we get

$$a_{n+1}(l) = \tfrac{1}{2}\{1 + \exp[-(2\pi/N)\mathrm{i}l]\} a_n(2l) . \quad (16)$$

For $l=0$ we have $a_{n+1}(0) = a_n(0)$, i.e. the operator (14) always has the eigenvalue 1, corresponding to conservation of the mean field at mixing. For $l \neq 0$ there are two possibilities.

(1) For some m the relation $2^m l = CN + \tfrac{1}{2}N$ holds with C integer (this can occur only for even N). Then $a_{n+m}(l)/a_n(2^m l) = 0$ because

$$1 + \exp[-(2\pi/N)\mathrm{i}(CN + \tfrac{1}{2}N)] = 0 .$$

(2) For some m, $2^m l = l + CN$, i.e. the sequence l, $2l$, $4l$, ... is periodic modulo N with period m. In this case

$$\frac{a_{n+m}(l)}{a_n(l)} = 2^{-m} \prod_{k=0}^{m-1} \{1 + \exp[-2^k(2\pi/n)\mathrm{i}l]\}$$

$$= 2^{-m} \frac{1 - \exp[-2^m(2\pi/N)\mathrm{i}l]}{1 - \exp[-(2\pi/N)\mathrm{i}l]} = 2^{-m} .$$

We conclude from this that there are three groups of eigenvalues:

type A: the eigenvalue 1 corresponding to mean field (zero wavenumber);

type B: the eigenvalues 0 corresponding to case 1 above;

type C: the eigenvalues with modulus equal to $\tfrac{1}{2}$ corresponding to case 2 above.

Comparing these properties with those of the continuous mixing operator, we conclude that eigenvalues with modulus equal to $\tfrac{1}{2}$ correspond to nonsmooth fields with $\beta = 1$, i.e. to discontinuous functions, while zero eigenvalues correspond to smooth fields. The number of eigenvalues of one or another type depends only the value of N. For $N = 2^m$

there are no type C eigenvalues, while for odd N there are no zero eigenvalues.

We may expect nontrivial dynamics in MML only if there is at least one type C eigenvalue. Then according to (13) a spatially homogeneous solution will be unstable if the Lyapunov number in the mapping f exceeds 2. Thus we choose for numerical simulations the following mapping,

$$\begin{aligned} f(u) &= (2+\epsilon)u\,, & 0<u<u^1 \equiv (2+\epsilon)^{-1}\,, \\ &= 1-(2+\epsilon)(u-u^1)\,, & u^1<u<u^2\equiv 2u^1\,, \\ &= (2+\epsilon)(u-u^2) & u^2<u<1\,, \end{aligned} \qquad (17)$$

The Lyapunov number of this piecewise linear mapping is $2+\epsilon$. Results are presented in figs. 1 and 2. In a wide range of wave numbers the spatial energy spectrum has a power law behaviour:

$$E(k) = \langle |a(k)|^2 \rangle \sim k^{-2}\,.$$

Similar spectra were observed in some CML models [7]. We also calculated the structure function scaling exponent $\zeta(p)$ defined as

$$\langle |u(x)-u(x+r)|^p \rangle \sim r^{\zeta(p)}\,,$$

see fig. 3. This exponent cannot be approximated by a straight line $\zeta(p) \simeq \gamma p$, so the field $u(x)$ appears to be multiaffine [8]. It should be noted, that in contrast to the CML models of ref. [7] the MML model discussed above is not Galilean invariant. In particular, the difference $|u(x)-u(x+r)|$ cannot be larger than 1 (the size of the interval where the mapping (17) is defined). Thus a good scaling like in figs. 1–3 is observed only for sufficiently small ϵ, when values of $|u(x)-u(x+r)|$ are much less than 1. For larger values of ϵ the power law spectrum saturates at small wavenumbers due to constraints of the mapping (17). The scaling properties of the fields generated by the MML will be discussed more thoroughly elsewhere.

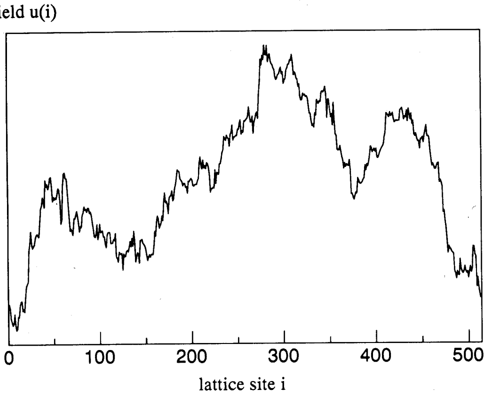

Fig. 1. Snapshot of the field resulting from MML (2), (14), (17) with $N=513$, $\epsilon=10^{-4}$ after 2×10^3 iterates starting from a random initial state.

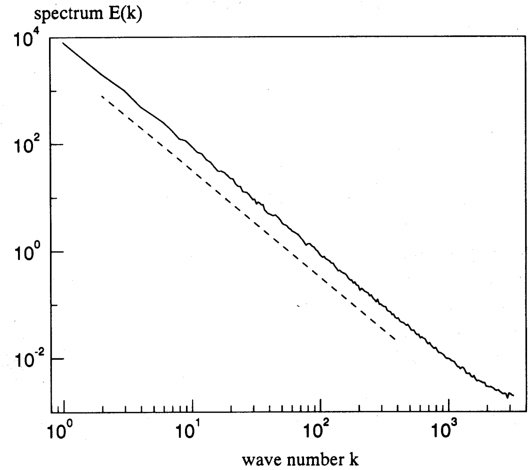

Fig. 2. Spatial power spectrum $E(k)$ on a log-log plot for $\epsilon=10^{-4}$, $N=6561$, averaged over 500 iterates. The slope of the dashed line is -2.

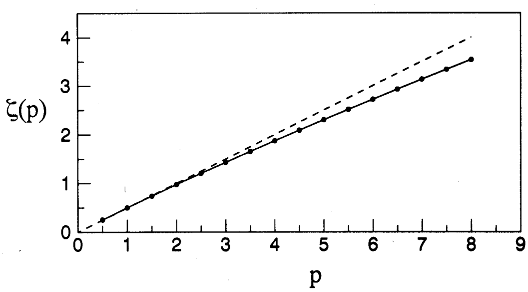

Fig. 3. Structure function $\zeta(p)$ for $\epsilon=10^{-4}$. Deviations from the dashed line $\zeta(p)=\frac{1}{2}p$ indicate multiaffinity.

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