Physics Letters A 165 (1992) 33–36
North-Holland

PHYSICS LETTERS A

Statistics of trajectory separation in noisy dynamical systems

Arkady S. Pikovsky [1,2]
Physics Department, Bergische Universität Wuppertal, Gaussstrasse 20, W-5600 Wuppertal 1, Germany

Received 11 June 1991; revised manuscript received 7 February 1992; accepted for publication 3 March 1992
Communicated by A.P. Fordy

Statistics of trajectory separation in a system of identical noisy mappings is investigated. We show that the probability density of the separation satisfies a scaling law whose exponent is determined by the spectrum of local Lyapunov exponents.

It is well known that a sensitive dependence on initial conditions is a criterion for chaotic behavior in deterministic systems [1]. Quantitatively, this sensitivity may be measured by the (maximal) Lyapunov exponent. A negative or zero Lyapunov exponent corresponds to regular motion, while a positive exponent corresponds to a chaotic one. The Lyapunov exponent may also be defined for noisy systems, i.e. dynamical systems explicitly governed by external random noise [2]. In this case the difference between systems with positive and negative Lyapunov exponents is not trivial. It is clear, that the qualitative behavior of a single system does not depend on the sign of the Lyapunov exponent: in both cases it is stochastic. However, the behavior of an ensemble of identical systems does depend drastically on the sign of the Lyapunov exponent [3]. Consider two systems with slightly different initial conditions, governed by the same noise. For a negative Lyapunov exponent the distance between trajectories of these systems in the phase space decreases, and eventually they become totally synchronized. For a positive Lyapunov exponent the distance between trajectories increases and they become desynchronized. The same arguments hold for an ensemble of systems with identical laws of motion and identical noise. This ensemble is represented by a cloud of points in the phase space. For a negative Lyapunov exponent this cloud shrinks eventually to a single point, while for a positive Lyapunov exponent it is distributed over the phase space. Thus, the sign of the Lyapunov exponent determines whether the systems will be synchronized or not. In ref. [3] the processes of synchronization and desynchronization were investigated for a noisy Zaslavsky map. Some models of this type were considered in refs. [4,5]. A similar effect of synchronization of subsystems, governed by the same chaotic signal, was recently described in ref. [6].

In ref. [5] the properties of noisy dynamical systems near the point of transition (where the Lyapunov exponent vanishes) were studied and very intermittent behavior was observed. In this paper we present a theory which gives a power-law distribution for trajectory separation near the transition point. The exponent is shown to depend on the spectrum of local Lyapunov exponents.

We consider a pair of one-dimensional noisy maps

$$x_{n+1}=f(x_n)+\xi_n\,,\quad y_{n+1}=f(y_n)+\xi_n\,, \tag{1}$$

where ξ_n are independent random variables. The separation $r_n=|x_n-y_n|$ is governed for small r by the linear equation

$$r_{n+1}=|f'(u_n)|\,r_n\,, \tag{2}$$

where $u_n=\frac{1}{2}(x_n+y_n)$. If we define the Lyapunov exponent λ as

$$\lambda=\langle\ln|f'(u)|\rangle=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\ln|f'(u_n)|\,, \tag{3}$$

[1] Alexander von Humboldt Fellow.
[2] On leave of absence from the Institute of Applied Physics, Russian Academy of Sciences, Gorky, Russia.

then it follows from eq. (2) that for $\lambda<0$ $r_n\to 0$ and for $\lambda>0$ $r_n\to\infty$. Let us introduce the variable $z_n=\ln r_n$, then we get

$$z_{n+1}=z_n+\ln|f'(u_n)| . \tag{4}$$

The increments $z_{n+1}-z_n$ are equal to "instant" Lyapunov exponents $\ln|f'(u_n)|$, they are random variables. Generally, one cannot consider them as independent, but we may expect that the correlations decay exponentially [#1]. Thus we may assume that the averaged quantities

$$\Lambda_k=\frac{1}{N}\sum_{n=kN}^{(k+1)N-1}\ln|f'(u_n)| \tag{5}$$

obey central limit theorem behavior for $N\to\infty$. This means that we may use for the probability distribution $p(\Lambda;\,N)$ of these "local" or "effective" Lyapunov exponents [7,8] an ansatz [9]

$$p(\Lambda;N)\sim\exp[-N\phi(\Lambda)] . \tag{6}$$

The scaling function $\phi(\Lambda)$ has a minimum just at $\Lambda=\lambda$ with $\phi(\Lambda)=0$. Near this minimum $\phi(\Lambda)$ may be approximated by a parabola

$$\phi(\Lambda)\approx C(\Lambda-\lambda)^2 ,\quad C>0 , \tag{7}$$

and substitution of (7) in (6) gives a Gaussian distribution. However, for large $|\Lambda-\lambda|$ the scaling function ϕ differs from (7), and this results in deviations from the Gaussian law for the tails of distribution (6) (see ref. [10] for a more detailed discussion of eqs. (6), (7)). For the generating function $G(s;\,N)$ of the distribution (6), defined as

$$G(s;N)=\int \mathrm{d}\Lambda\, p(\Lambda;N)\,\exp(sN\Lambda) ,$$

we have

$$G(s;N)\sim\int \mathrm{d}\Lambda\,\exp\{N[s\Lambda-\Phi(\Lambda)]\} .$$

For $N\to\infty$ this integral is dominated by the maximum of the function $s\Lambda-\phi(\Lambda)$ and we obtain [11]

$$G(s;N)\sim\exp[Ng(s)] , \tag{8}$$

where $g(s)$ is defined from

$$g(s)=s\Lambda-\phi(\Lambda),\quad s=\mathrm{d}\phi/\mathrm{d}\Lambda . \tag{9}$$

[#1] This is the reason for considering *noisy* systems: in a deterministic system the correlation time diverges near the point of transition to chaos.

Relation (9) is the Legendre transformation which is well known in statistical mechanics [12]. For small s we are near the minimum of $\phi(\Lambda)$ and substituting (7) into (9) we obtain

$$g(s)\approx s\lambda+s^2/4C . \tag{10}$$

We will now show that eqs. (4)–(10) give a power law for the distribution of r. We obtain from eq. (4)

$$z_{(k+1)N}=z_{kN}+N\Lambda_k . \tag{11}$$

Here z_{kN} and Λ_k may be considered as independent variables. It is known that the probability distribution of a sum of independent random variables equals the convolution of the probability distributions of these variables. Thus for the evolution of the probability distribution $W_k(z)$ we get

$$W_{k+1}(z)=\int \mathrm{d}\Lambda\, p(\Lambda;N)\,W_k(z-N\Lambda) . \tag{12}$$

Looking for a stationary distribution in the form

$$W_k(z)=W_{k+1}(z)\sim\exp(\sigma z) , \tag{13}$$

we get

$$1=\int \mathrm{d}\Lambda\, p(\Lambda;N)\,\exp(-N\sigma\Lambda)$$

$$\approx\exp[Ng(-\sigma)] . \tag{14}$$

Thus, for r this gives a power law probability density

$$P(r)\sim r^{\sigma-1} , \tag{15}$$

with σ satisfying (taking the limit $N\to\infty$ in (14))

$$g(-\sigma)=0 . \tag{16}$$

Eq. (16) has two solutions, one of them, $\sigma=0$, is of no physical meaning because it corresponds to a uniform distribution, which always satisfies (12). In the approximation (10) the nontrivial solution of (16) is

$$\sigma\approx 4C\lambda \tag{17}$$

and it changes sign exactly at $\lambda=0$. (Because s is assumed to be small in (10), eq. (17) is valid for small λ only.) Thus, the distribution (15) is non-normalizable (as any perfect power law) and its integral diverges for $r\to 0$ when $\lambda<0$ and for $r\to\infty$ when $\lambda>0$.

In order to obtain a normalizable distribution we have to include additional terms in eq. (2). For small r such terms arise if we take into account a slight breaking of the $x \leftrightarrow y$ symmetry of eq. (1). Let us assume that the mapping f depends on a parameter a, and consider two mappings with slightly different parameters $a \pm \alpha$:

$$x_{n+1} = f(x_n, a+\alpha) + \xi_n ,$$
$$y_{n+1} = f(y_n, a-\alpha) + \xi_n . \quad (18)$$

Then instead of eq. (2) we obtain

$$r_{n+1} = | \pm f'_u(u_n, a) r_n + 2 f'_a(u_n, a)\alpha | . \quad (19)$$

For $r < \alpha$ the second term in (19) dominates, thus giving a cutoff of the probability distribution (15) at $r \approx \alpha$. A similar cutoff is provided if the noise for variables x and y is slightly different. For $r \gg \alpha$ the second term in eq. (19) may be neglected, in this region the power law (15) is observed.

For large r it is clear that nonlinear terms should be added to eq. (2) in order to obtain a normalizable distribution. These terms depend on the precise form of the mapping f. As a crude approximation we may simply cut off the power law (15) at the system's phase space size r_0 (r_0 may be roughly defined as the maximal distance in the phase space between the points on the attractor) and use the following model (as one can see from fig. 1 below, this approximation is not bad),

$$P(r) = \sigma r_0^{-\sigma} r^{\sigma-1} \quad \text{for } r \leqslant r_0 ,$$
$$= 0 \quad \text{for } r > r_0 , \quad (20)$$

where r_0 is the size of the system's phase space. From (20) one easily obtains the moments of r,

$$\langle r^q \rangle = \frac{\sigma}{q+\sigma} r_0^q . \quad (21)$$

Substituting (17) into (21) we obtain for $|\lambda| \to 0$

$$\langle r^q \rangle \sim \lambda \times 4 C q^{-1} r_0^q . \quad (22)$$

Consider now not a pair of identical systems, but a large ensemble,

$$x^i_{n+1} = f(x^i_n) + \xi_n, \quad i = 1, 2, ..., M . \quad (23)$$

If we suppose that all x^i_n are close to each other, the averaged separation D_n, defined as

$$D_n = \left(\frac{1}{M} \sum_1^M (x^i_n - u_n)^2 \right)^{1/2} , \quad u_n = \frac{1}{M} \sum_1^M x^i_n , \quad (24)$$

obeys the equation

$$D_{n+1} = |f'(u_n)| D_n , \quad (25)$$

which coincides with eq. (2). Thus the statistics of D is the same as that of r. In ref. [5] it was shown that $\langle D \rangle \sim \lambda$. Eqs. (21) and (22) generalize this result.

We checked the main result of this paper – eqs. (15), (16) – with the following piecewise-linear mapping,

$$x_{n+1} = f(x_n) + \xi_n \ (\text{mod } 1) , \quad (26)$$

where ξ_n is random noise uniformly distributed between 0 and 1, and

$$f(x) = ax \quad \text{for } x < \tfrac{1}{3} ,$$
$$= \tfrac{1}{3} a + (3-2a)(x - \tfrac{1}{3}) \quad \text{for } \tfrac{1}{3} \leqslant x \leqslant \tfrac{2}{3} ,$$
$$= 1 - a(1-x) \quad \text{for } x > \tfrac{2}{3} . \quad (27)$$

Due to the noise, the invariant measure of this mapping is uniform independently of $f(x)$ and successive values of x_n are independent. Thus the Lyapunov exponent λ is simply related to a: $\lambda = \frac{2}{3} \ln a + \frac{1}{3} \ln |2a-3|$ and eq. (12) becomes exact even for $N=1$. (It should be noted that without noise the mapping (26), (27) has a stable fixed point for $1 < a < 2$, while with noise λ changes sign for $a \approx 1.7$; this shows that the Lyapunov exponent can change drastically when noise is added.) Taking into account that

$$|f'(x)| = a \quad \text{for } 0 < x < \tfrac{1}{3}, \tfrac{2}{3} < x < 1 ,$$
$$= |3-2a| \quad \text{for } \tfrac{1}{3} \leqslant x \leqslant \tfrac{2}{3} ,$$

we may write the probability distribution function $p(\Lambda; 1)$ as a sum of two Dirac δ-functions:

$$p(\Lambda; 1) = \tfrac{2}{3} \delta(\Lambda - \ln a) + \tfrac{1}{3} \delta(\Lambda - \ln |2a-3|) .$$

Substituting this expression and $N=1$ in (14) we get

$$1 = \int d\Lambda \, p(\Lambda; 1) \exp(-\sigma \Lambda)$$
$$= \tfrac{2}{3} \exp(-\sigma \ln a) + \tfrac{1}{3} \exp(-\sigma \ln |3-2a|)$$
$$= \tfrac{2}{3} a^{-\sigma} + \tfrac{1}{3} |2a-3|^{-\sigma} . \quad (28)$$

Numerically obtained probability densities and the exponents σ are presented in figs. 1 and 2.

In fig. 2, the exponents obtained are both from the separation between trajectory pairs (crosses) and from the spreading of the trajectory cluster according to eqs. (24), (25). For the former, one can see

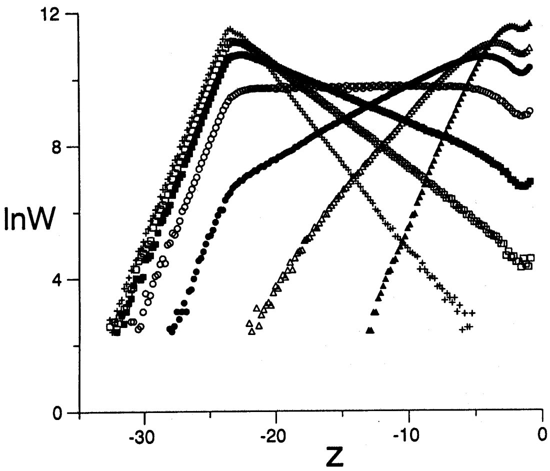

Fig. 1. Probability density (in arbitrary units) of the variable z in the system defined by eqs. (15), (27) for $\alpha=10^{-10}$ and different values of λ: (+) $\lambda=-0.2$, (□) $\lambda=-0.1$, (■) $\lambda=-0.05$, (○) $\lambda=0$, (●) $\lambda=0.05$, (△) $\lambda=0.1$, (▲) $\lambda=0.25$. Between the low-value cutoff at $z_l \approx \ln \alpha \approx -23$ and the high-value cutoff at $z \approx -3$ one observes a linear dependence of $\ln W$ on z. The slope gives the exponent σ.

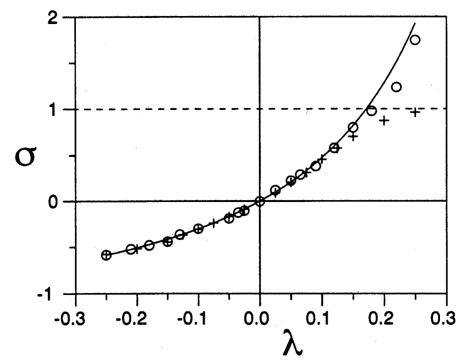

Fig. 2. Slopes of the graphs of fig. 1 between low- and high-z cutoffs. (+) r-statistics; (○) D-statistics; (——) theory, eq. (28).

a clear violation of eq. (16) near $\sigma=1$. This can be understood if we go beyond the linear approximation. Indeed, eq. (2) describes linear trajectory separation from small to large r. This equation should be complemented by nonlinear saturation for large r and "reinjection" for small r. Saturation means that when r is large, trajectories x and y become statistically independent and their joint probability distribution is

$$w(x,y)=w_0(x)w_0(y) . \tag{29}$$

"Reinjection" to small r occurs when these statistically independent trajectories come occasionally very close to each other. The probability distribution for reinjecting small r may be estimated from (29),

$$P_r(r)\,dr=\text{prob}(r<|x-y|<r+dr)$$
$$=\text{const}\,dr , \tag{30}$$

where const$=2^{3/2}\int w_0^2(x)\,dx$. What is in fact observed is a mixture of power law (15) with the constant reinjection distribution (30). For $\sigma<1$ the power law dominates for small r, while for $\sigma>1$ the constant distribution is observed in accordance with fig. 2. For an ensemble of M particles similar arguments show that $P_r \sim r^{M-2}$. This explains why power law (15) holds for large ensambles even for $\sigma>1$.

In conclusion, we would like to mention that the essential part of the analysis presented here is based on the fact that the evolution of the trajectory separation is governed by a local Lyapunov exponent. The same occurs in coupled chaotic attractors, where a similar power-law behavior of the asymmetrical component was recently described [13].

The author thanks the Alexander von Humboldt Stiftung for support and P. Grassberger for valuable discussions.

References

[1] H.G. Schuster, Deterministic chaos: an introduction (Physik Verlag, Weinheim, 1988).
[2] T. Kapitaniak, Chaos in systems with noise (World Scientific, Singapore, 1988).
[3] A.S. Pikovsky, Radiophys. Quantum Electron. 27 (1984) 576; in: Nonlinear and turbulent processes in physics, Vol. 3, ed. R.Z. Sagdeev (Harwood, New York, 1984) p. 1601.
[4] F.J. Romerias, C. Grebogi and E. Ott, Phys. Rev. A 41 (1990) 784.
[5] L. Yu, E. Ott and Q. Chen, Phys. Rev. Lett. 65 (1990) 2935.
[6] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. 64 (1990) 821.
[7] H. Fujisaka, Prog. Theor. Phys. 70 (1983) 1264.
[8] P. Grassberger and I. Procaccia, Physica D 13 (1984) 34.
[9] J.-P. Eckmann and I. Procaccia, Phys. Rev. A 34 (1986) 659.
[10] P. Grassberger, R. Badii and A. Politi, J. Stat. Phys. 51 (1988) 135.
[11] P. Grassberger, in: Chaos, ed. A.V. Holden (Manchester Univ. Press, Manchester, 1986) p. 291.
[12] R. Kubo, Statistical mechanics (North-Holland, Amsterdam, 1965).
[13] A.S. Pilovsky and P. Grassberger, J. Phys. A. 24 (1991) 4587.