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Farey level separation in mode-locking structure of circle mappings

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We consider various characteristic scales both on the X-axis and in the parameter space for critical and near-critical circle mappings. Juxtaposition of different scales for the set of all rational mode-lockings exposes the effect of separation into layers. This layered structure is shown to reproduce Farey-tree levels of the corresponding rotation numbers. The separation is studied numerically for maps with various orders of singularity and interpreted in terms of coefficients of rotation function expansion.

1. Introduction

One-dimensional circle mappings have become the convenient tool for studying the phenomena associated with transitions to chaos in dissipative systems with two fundamental frequencies (for references and applications see, e.g., the recent review [1]). The typical (and most widely studied) representative of these maps is the "sine" family

$$x_{i+1} = f(x_i) = x_i + \Omega - \frac{K}{2\pi}\sin(2\pi x_i)\,, \qquad (1)$$

where the parameter K governs the nonlinear interaction of modes. The subcritical maps (in the family (1) they are those with $K<1$) are diffeomorphisms whereas the supercritical ones ($K>1$) are non-invertable and may display chaotic dynamics. The borderline between the two cases comprises critical mappings – homeomorphisms with singularities (which are commonly cubic inflection points); this corresponds to $K=1$ in (1).

The most widely used characteristics of dynamics of a circle mapping is its rotation number $\rho(x) = \lim_{n\to\infty}[f^n(x) - x]/n$ [2]. For subcritical and critical maps this number does not depend on the initial point x. The graph of the function $\rho(\Omega)$ is the so-called devil's staircase in which every rational value of the function corresponds to the "stair" – the interval of argument values.

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In these intervals the system is mode-locked – for rational $\rho = p/q$ one observes a q-periodic orbit. The Cantor set of parameter values which remains when all the mode-locked intervals of Ω are deleted, corresponds to irrational rotation numbers. This set is known to have positive measure in the subcritical case and zero measure in the critical [3]. We will concentrate below mainly upon the critical mappings.

The properties of critical families may be roughly divided into local and global ones. The former refer to the vicinities of fixed values of ρ and the latter characterize the entire devil's staircase or its large segments. It is well known that in the parameter space the devil's staircase exhibits both the local and global self-similarity. In the former case this is the exponential scaling of mode-locked intervals corresponding to rational approximations of some irrational numbers (of which the inverse golden mean $\sigma = \frac{1}{2}(\sqrt{5} - 1)$ is the best known [4,5]). As for the global scaling the most striking feature seems to be the universality of the fractal dimension of the set of parameter values related to irrational rotation numbers. This dimension is the same not only over the entire staircase but also for all maps with the same order of inflection [6,7].

On the X-axis one also observes local (in the above sense) scaling behavior: the distance between the inflection point and the nearest point of the superstable trajectory decreases exponentially as we bypass rational approximants to the fixed irrational rotational number. The exponents are determined by the structure of the expansion of ρ into the continuous fraction [4,5].

Local scaling behavior means that there exists quantitative interrelation between individual mode-locking intervals: if one knows the parameters for several first intervals, then it is possible to predict with high accuracy individual characteristics of all other intervals belonging to the same structure. For global scaling, as far as we know, only averaged quantities (such as fractal dimension or scaling spectrum) were calculated, and this does not allow to make predictions about parameters of individual mode-locking intervals.

In this paper we consider individual characteristics of mode-locked intervals. Our approach is to juxtapose in the parameter space and on the X-axis the scaling phenomena which accompany the descent down the mode-locking structure for all rational rotation numbers. Numerical data for this juxtaposition clearly demonstrate the separation of characteristic sizes of mode-lockings according to the Farey level of their rotation numbers. We discuss also the connection of this phenomenon with the order of singularity of critical maps. Finally the separation observed is formulated in terms of coefficients of rotation functions in "centroids" of critical mode-lockings.

2. Characteristic scales of mode-lockings

Before the computation we must agree on the number and definitions of characteristic scales which should be appropriate for all mode-lockings. The existing results imply that two parameters are necessary (and sufficient): for the scales along the critical line (i.e., in the subspace of critical mappings) and across it. Take the rational rotation number p/q. For the critical mappings the choice in the parameter space is unambiguous: let Ω_ℓ and Ω_r be the left and the right ends of the mode-locked interval on the critical line (fig. 1). Then $\omega = \Omega_\mathrm{r} - \Omega_\ell$ is the most natural size scale along this line.

Another characteristic length in the parameter space should describe the direction transversal to the critical line – this corresponds to the transition from subcritical to supercritical mappings. This scale may be introduced in different ways. In the supercritical domain one encounters period-doublings and chaos. We may take either k_1 – the shortest distance between the critical line and the first period-doubling bifurcation, or k_2 – the distance which separates the critical line

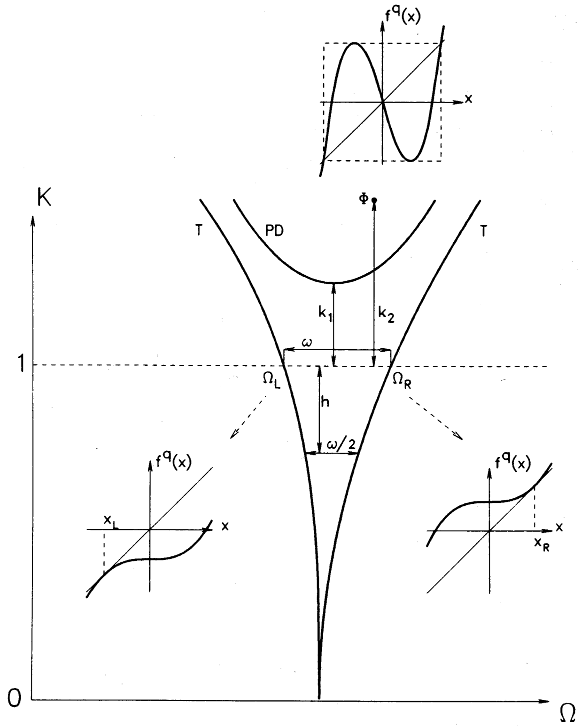

Fig. 1. Characteristic scales of p/q-locking. T – lines of tangent bifurcation; PD – line of period-doubling bifurcation; Φ – point of "full" chaos (see the text).

from the point of "full chaos" – the point Φ in which the circle is decomposed by f^q into q invariant intervals each of the latter being mapped onto itself threefold (fig. 1). In the subcritical domain all the locked regions have the same length – from $K=1$ to $K=0$. To characterize the difference between these regions near the critical line one may use the depth h – the level of subcriticality at which the width of the locked region (Arnold's tongue) is half of the critical $\Omega_r - \Omega_\ell$. Our numerical data point to the fast decrease of h with growth of q; except for the numbers of the kind $1/N$ and $1-1/N$ (for which the decrease is slower) h is less then 0.05 for $q>20$.

The situation for the X-axis is less clear since, strictly speaking, not the values of characteristic sizes but the whole scaling function along the orbit should be considered. For simplicity we shall take only one number for each mode-locking – so to say, the most characteristic length, but even here various definitions are possible. We do it as follows. In the critical case (as distinct from subcritical) there is one specific point on the circle – the inflection point of the function f. In the endpoints of the locked interval the tangent bifurcations take place – stable periodic orbits collide with unstable ones. We take $d=|x_\ell x_r|^{1/2}$ where x_ℓ and x_r are the distances between the inflection and the nearest point of this neutral (semi-stable) orbit for Ω_ℓ and Ω_r respectively (fig. 1). One may take another choice: the length of the largest arc on the circle between two neighboring points of neutral q-periodic orbit (either for Ω_ℓ or for Ω_r or the average value). For most of the p/q the inflection point belongs to this interval (which is quite natural since near the cubic inflection the graph of the function is nearly horizontal) and the whole scaling picture is practically the same.

3. Farey levels in distribution of characteristic scales

Now for any rational $\rho=p/q$ we have a set of numbers: d on the X-axis, ω on the Ω-axis and k_1 (or k_2 or h) on the K-axis. First we will restrict ourselves to the dependence between d and ω. In fig. 2a in logarithmic coordinates $v=\log(d)$ and $u=\log(\omega)$ the data are presented for 255 various mode-lockings in the interval $\frac{1}{2}<\rho<1$ for the family (1) with $K=1$. Each p/q is represented by a point. It is easy to see that these points do not fill the plane uniformly but display the evident tendency to group into layers – almost parallel stretched clusters separated by practically equal blank strips. The counting shows that each layer comprises twice as many points as its upper neighbor. It is but natural to look what is common for all the lockings whose points belong to the same layer. Our numerical observations allow us to formu-

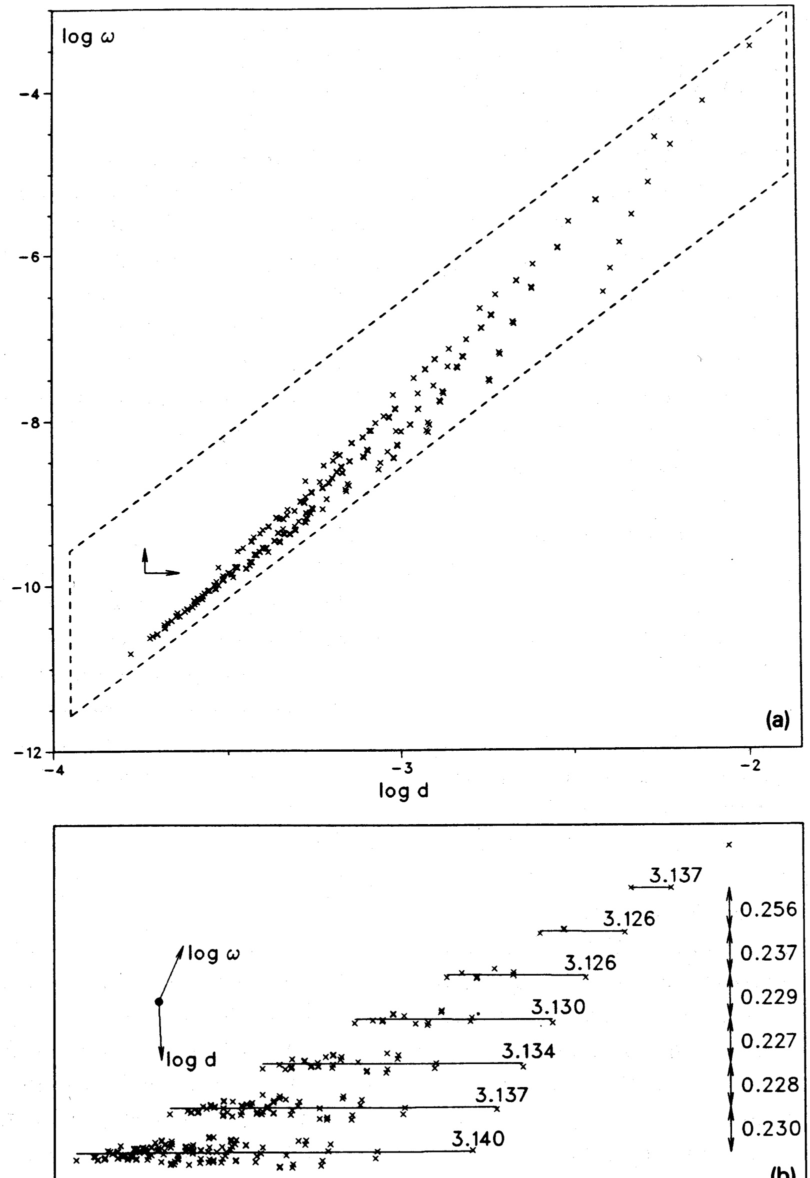

Fig. 2. Dependence of log(ω) on log(d) for 255 critical lockings in the family (1).

late the following conjecture (which is, in fact, the main statement of this paper):

Conjecture. Rational numbers related to the points of each layer constitute exactly one level of the Farey tree, and, conversely, in the chosen coordinates each Farey level is represented by the layer.

The Farey numbers arise most naturally in the context of various phenomena associated with ordered mode-lockings [7–11]. The Farey tree is a result of multiple repetition of the well-known Farey addition: $p_1/q_1 \oplus p_2/q_2 = (p_1 + p_2)/(q_1 + q_2)$. Given the two first-level rationals 0/1 and 1/1 we produce the second level number 1/2 as $(0+1)/(1+1)$ and repeat this process. To construct each new Farey level we must insert between each two already existing neighboring rationals a new one which is generated by their Farey addition. A natural question is what is in common for the numbers in the same level. It can be shown that the nth ($n>2$) level of the Farey tree consists of those (and only those) 2^{n-2} rationals whose sums of entries in the continuous fraction representation $\rho = 1/[m_1 + 1/(m_2 + \cdots 1/m_k)\cdots] = [m_1, m_2, \ldots, m_k]$ obey the condition $\Sigma_i^k m_i = n$. As one proceeds to the next level the sum grows by unit. Thus of all the irrational values of ρ only the inverse golden mean $\sigma = [1, 1, \ldots, 1, \ldots]$ is represented by its rational approximations in every level of the Farey tree whereas the approximations to the numbers with large entries belong to strongly separated levels. It should be remembered that the vicinities of rationals with large entries close to $1/N$ and $1-1/N$ fail to exhibit universal scaling properties [8].

For convenience the set of points from fig. 2a is presented in fig. 2b in the oblique-angled coordinate system in which the last layer is horizontal; directions of the initial coordinate axes are shown by arrows. The location of straight lines passing through each layer is determined by the least squares method. The number upon each line gives its slope (in the initial coordinates of fig. 2a). The distances between the lines are also shown. These data strengthen our visual impression that the layers are nearly parallel and equidistant.

The figure presents only eight of the Farey levels (from the third to the tenth). We performed the computations for all the rationals from 13 levels (8191 mode-lockings) and at each new level the picture was qualitatively and quantitatively the same. Naturally in each layer there are some splashes but the bulk of the points adjoins its axis thus stabilizing the average characteristics. For $M = 2^{j-2}$ points of the jth layer with the use of the standard formulae

$$\bar{y} = M^{-1} \sum_{i=1}^{M} y_i \,, \quad D_y = M^{-1} \sum_{i=1}^{M} (y_i - \bar{y})^2 \,,$$

$$Q_{yz} = (D_y D_z)^{-1/2} M^{-1} \sum_{i=1}^{M} (y_i - \bar{y})(z_i - \bar{z}) \,, \tag{2}$$

the mean values $\bar{v}^{(j)}$ and $\bar{u}^{(j)}$, variances $D_v^{(j)}$ and $D_u^{(j)}$ and the correlation coefficient $Q_{uv}^{(j)}$ were computed. For 13 studied layers $0.998 < Q_{uv}^{(j)} < 1$ which permits to conclude that the statistical dependence between $u^{(j)}$ and $v^{(j)}$ is very close to linear. Therefore the coordinates u and v of the point belonging to the jth Farey level are statistically coupled through the relation

$$u \approx C_0 - bj + \kappa v \,, \tag{3}$$

where $\kappa = 3.148\ldots$, $b = 0.2369\ldots$ (the values are obtained from the higher levels of the Farey tree, hence their slight decline from the data in fig. 2b) and C_0 is a dimensionalization factor. The data show a small difference between the estimate of κ and familiar value of π.

Then for the characteristic scales ω and d one observes

$$\omega \approx C_1 B^{-j} d^{\kappa} \,, \tag{4}$$

where $C_1 = \exp(C_0)$, $B = \exp(b) = 1.267\ldots$.

The expressions (3) and (4) surely deserve some discussion. Being obtained from fitting the numerical data, they do not pretend to have overall validity. The plot yields a striking evidence that at least for a few hundreds of the first rationals in the Farey scheme the separation action is clearly visible and the characteristics of the lockings are rather close to the layer axes described by (3). Notwithstanding this convincing graphical effect, counterexamples in the higher levels of the Farey tree are not very hard to find. To see better, one is to proceed from individual mode-lockings to families of lockings, convenient examples being the sequences of the extreme points from each layer.

In each layer the leftmost point is the corresponding rational approximation to the golden mean. The slope of the line passing through these points equals $\log\delta/\log(\alpha) = 4.1078\ldots$, where $\alpha = 1.2886\ldots$ and $\delta = 2.8336\ldots$ are the local scaling factors near σ on the X-axis and in the parameter space respectively [4,5]. Analogously one may trace the other lines, which lie across the layer system and correspond to other irrational rotation numbers with eventually periodic expansion into the continuous fraction. The slopes of these lines are, likewise, ratios of the logarithms of the local scaling factors near these irrational numbers. To ensure that the rational approximants do not miss the layers, the distances between them in both ω- and X-directions (these being, naturally, δ and α for the corresponding irrational ρ) should be commensurate with characteristics of the layered structure. This leads to the condition $\delta \cong B^m \alpha^\kappa$, where m is the sum of the entries over the period of the continuous fraction. Substituting the values for δ and α obtained from the renormalization analysis, we see that both for the golden mean ($m = 1$) and $\frac{1}{2}(\sqrt{2} - 1) = [\ldots, 2, 2, 2, \ldots]$ ($m = 2$) this condition is fulfilled within 1%. The numbers to follow are those with $m = 3$: $[\ldots, 1, 2, 1, 2, \ldots]$ and $[\ldots, 3, 3, 3, \ldots]$, for which the condition is matched within 3%. Hence follows that whenever one of the points in a sequence is located near the axis of some layer, all the following points are close to axes of their respective layers. We may conclude that the approximations to these irrationals (which dominate in the first levels of the Farey tree, since the numbers with large entries are rare guests there) easily find their places in the layers and obey the laws like (3). We also notice here that the considered effect is closely connected to the problem of distribution of eigenvalues δ and scaling factors α for the fixed points of renormalization operators, associated to different irrational rotation numbers.

The situation with the rightmost points of the layers does not seem to be so good. These points correspond to the "harmonic" numbers $1 - 1/(j+1)$. The locked intervals ω for the harmonic sequence are known to scale as j^{-3} [3,8,12], which is slower than the mentioned exponential scaling of approximants to irrational numbers. To satisfy (4) and counterbalance the influence of exponentially decreasing factor B^{-j} the characteristic size d should be eventually *growing*. Hence the harmonic numbers might not in principle obey the laws like (4). However, due to the fact that B is close to 1, whereas d remains finite and bounded away from zero [3,8] (this can be also seen in fig. 2a, where the rightmost points tend to a vertical asymptote) the difference between the power law and the exponent is not strong enough to influence the first Farey levels. One may see that for the plotted layers the harmonic numbers are remarkably well described by (3); considerable discrepancies are expected to appear only at $j \cong 15$. The decline from (4) in harmonic-like sequences which are nested between every two neighbours in the Farey tree will be observable respectively much later. Moreover, the purely combinatorial arguments show that in the Farey-ordering the harmonic-like sequences are relatively rare and make only modest contribution to the statistical averages (2), whereas those with exponential scaling (and moderate entries in continuous fractions) prevail in all the levels. So far as we are treating the laws (3), (4) as empirical, we should not expect them to be uniformly valid. At the same time, for the applications, where typically only the relatively simple mode-lockings, corresponding to rotation numbers from the first Farey levels, can be instrumentally resolved, these laws (as well as the similar ones for transcritical scales, discussed below) might be useful, allowing to estimate all the characteristic scales of the locking from the value of only one of them.

The question now is whether the relations (3), (4) are universal both qualitatively and quantitatively. To check this we have performed the calculations for 4095 rational values of ρ in the two-parameter family

$$x_{i+1} = f(x_i) = x_i + \Omega - \frac{1+3U}{2\pi}\sin(2\pi x_i) + \frac{U}{2\pi}\sin(6\pi x_i)\,, \qquad (5)$$

which is critical for $-\frac{1}{3} < U < \frac{1}{24}$. The results demonstrate the same separation over the Farey tree. As for the slope of layers κ and the distance b between them, some weak dependence on U is detected. It can be seen from tables 1, 2 that the convergence to limit values with the increase of j (the number of the Farey level) is rather slow. Anyway the prospect of tending to the same universal limits seems at least doubtful and we suppose that both κ and b are functions of U which vary within the range $3.07 < \kappa < 3.26$, $0.227 < b < 0.245$ (for $U = \frac{1}{24}$ when the cubic term in the expansion of f vanishes and the inflection is of the fifth order, $\kappa = 4.628\ldots$ and $b = 0.289\ldots$; near this point the pronounced crossover phenomena are observed).

At the same time the increments from level to the next level of the averaged values $\bar{u}$ and $\bar{v}$ for large j are practically independent from U and seem for $j \to \infty$ to be universal for all mappings with cubic inflection points:

$$\bar{u}^{(j)} - \bar{u}^{(j+1)} \Rightarrow \gamma = 0.8718\,,$$
$$\bar{v}^{(j)} - \bar{v}^{(j+1)} \Rightarrow \chi = 0.2012\ldots. \qquad (6)$$

Table 1
Slopes of layers $\kappa^{(j)}$ for the critical family (3).

Number of the layer	$\kappa^{(j)}$						
	$U=-\frac{1}{5}$	$U=-\frac{1}{6}$	$U=-\frac{1}{10}$	$U=-\frac{1}{16}$	$U=-\frac{1}{40}$	$U=0$	$U=\frac{1}{40}$
$j=3$	5.5341	5.7486	4.1060	3.2108	2.9978	3.1371	3.7635
$j=4$	4.2665	5.4829	4.3493	3.4617	3.0907	3.1265	3.6609
$j=5$	2.3356	3.1889	3.9712	3.4937	3.1588	3.1264	3.5373
$j=6$	2.3039	3.0955	3.7712	3.4897	3.2036	3.1297	3.4206
$j=7$	2.6608	3.1599	3.6456	3.4650	3.2276	3.1338	3.3208
$j=8$	2.7958	3.1856	3.5567	3.4320	3.2373	3.1374	3.2405
$j=9$	2.8977	3.2024	3.4911	3.3991	3.2387	3.1403	3.1793
$j=10$	2.9481	3.2130	3.4420	3.3699	3.2358	3.1424	3.1348
$j=11$	2.9839	3.2169	3.4046	3.3453	3.2312	3.1440	3.1041
$j=12$	3.0072	3.2167	3.3755	3.3249	3.2259	3.1452	3.0840
$j=13$	3.0243	3.2144	3.3523	3.3081	3.2207	3.1460	3.0719

Table 2
Distances between layers for the critical family (3).

Number of the layer	Distance between jth and $(j-1)$th layers						
	$U=-\frac{1}{5}$	$U=-\frac{1}{6}$	$U=-\frac{1}{10}$	$U=-\frac{1}{16}$	$U=-\frac{1}{40}$	$U=0$	$U=\frac{1}{40}$
$j=4$	1.3697	0.1712	−0.2410	−0.0396	0.1586	0.2369	0.2721
$j=5$	2.2527	2.2929	0.3313	0.0959	0.1576	0.2368	0.3078
$j=6$	0.4485	0.3427	0.2910	0.1436	0.1662	0.2355	0.3308
$j=7$	0.0168	0.1364	0.2781	0.1859	0.1824	0.2347	0.3415
$j=8$	0.0910	0.1848	0.2754	0.2160	0.1990	0.2346	0.3422
$j=9$	0.1036	0.1925	0.2719	0.2333	0.2123	0.2348	0.3355
$j=10$	0.1766	0.2003	0.2668	0.2416	0.2217	0.2353	0.3242
$j=11$	0.1917	0.2131	0.2614	0.2448	0.2280	0.2358	0.3109
$j=12$	0.2103	0.2228	0.2569	0.2456	0.2320	0.2363	0.2973
$j=13$	0.2182	0.2294	0.2553	0.2453	0.2346	0.2369	0.2847

This means that the geometric mean length of the critical mode-locked region in the high Farey level is by a factor of $\exp(\gamma) = 2.391$ shorter than in the preceding level; the geometric mean for the X-scale is $\exp(\chi) = 1.222$ times smaller. Unexpectedly the value of γ is within numerical accuracy indistinguishable from the known estimates of the fractal dimension of the devil's staircase [6,7,10]. We must note that this coincidence takes place for the maps with cubic inflection points only (cf. fig. 3).

The limit γ (when it exists) is in fact nothing, but $\log(2)/D_i$, where D_i is the so-called information dimension of the mode-locking structure [10] (with the measure on the parameter interval introduced in accordance with the Farey tree scheme)[#1]. The more puzzling seems this sudden numerical interrelation of two different dimensions of the devil's staircase in case of cubic maps.

Substituting (6) into (3) we arrive at

$$\gamma \approx \kappa\chi + b\,, \tag{7}$$

which relates the average scaling factors γ and χ to the slope and distance of the layered structure and is consistent with the numerical data.

[#1] We would like to thank Erik Aurell for bringing this to our attention.

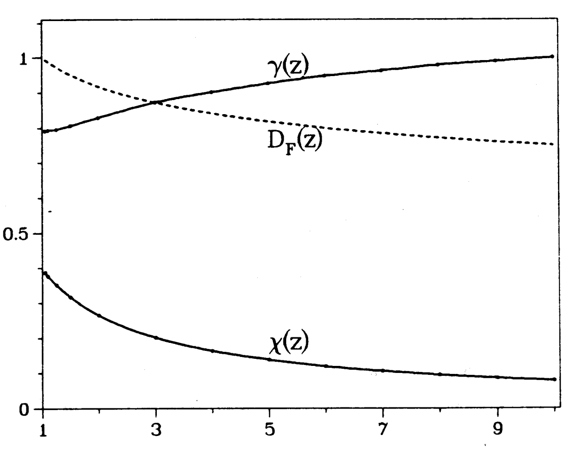

Fig. 3. Dependence of global characteristics of mode-locking structure on the singularity order z. $D_F(z)$ – fractal dimension of the devil's staircase.

As for the inner structure of the layers, some self-similar clusters may be detected in the layers related to high Farey levels. The order in which the mode-lockings fill in the corresponding layer seems to be in some sense universal, but this universality is beyond the scope of our current investigation (see [9] for some description). It should be also noted that inside each layer the characteristic scales decrease strictly monotonously with the growth of the denominator q of the rotation number ρ. The numerator p means less: the layers consist of subclusters, each one comprising all the lockings corresponding to rotation numbers with the same q (and, of course, belonging to the same Farey level), the inner distances within these subclusters being small compared with distances between them. The subclusters are oriented transversely to the layer axes; their edges stretch out into the interlayer blank space. For the higher Farey levels the largest clusters start reaching the adjacent layers and the whole structure becomes more fuzzy. However, the proportion of these "remote" lockings compared to the number of lockings lying close to the axis remain rather small.

4. Effects of the order of singularity

In our numerical studies we have found the Farey-layered separation not only in families with cubic inflections but also for the circle maps with singularities of the kind $x|x^{z-1}|$ $(1 \le z \le 10)$. The quantitative characteristics prove to be monotone functions of z: the growth of z leads to decrease of χ and increase of κ, b, γ (fig. 3). The distributions similar to that of fig. 2 for the same 255 rotation numbers from the 10 first Farey levels and some values of z are presented in figs. 4a, 4b. It can be easily seen that the higher the order of singularity is the more pronounced looks the layered structure. For $z \le 1.5$ the distance between the layers becomes smaller

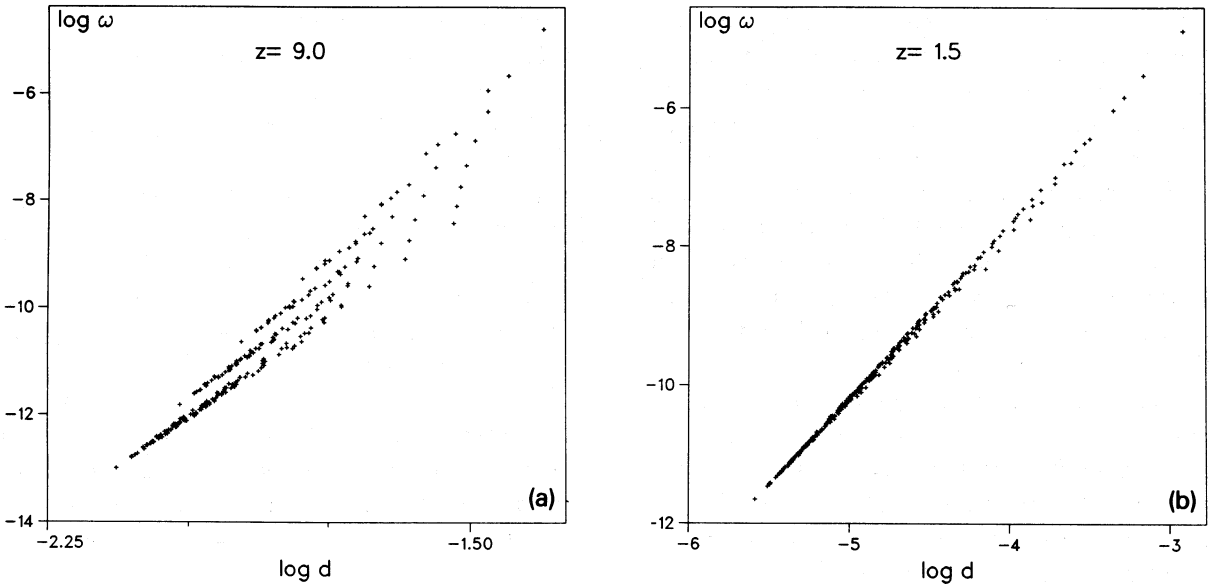

Fig. 4. Farey separation for different z: (a) $z=9$; (b) $z=1.5$.

than the transversal dispersion of points in high levels, this concealing the visual effect of separation.

As one gets closer to $z=1$ we arrive in the situation when there are no distinguished places on the circle and the points of the periodic orbit are distributed practically equidistantly. Therefore the characteristic size in X-space tends to q^{-1} (the inverse denominator of the rotation number). It can be easily seen that after each transition to the next Farey level the arithmetic mean (over the level) denominator is increased by a factor of $\frac{3}{2}$. The geometric mean denominator grows a bit slower with an increment tending to $1.48618\ldots=\exp(0.3962)\ldots$). This is just what we see in fig. 3 where $\chi(1)\cong 0.4$.

Maybe less expected is the numerical evidence that for $z\to 1$ one observes $\kappa(z)\to 2.0$ and $\gamma(z)\to 2\chi(1)$. This implies that the average width of the locked interval ω scales as q^{-2}. The seeming discrepancy with a well-recognized $\omega\cong q^{-3}$ (widely demonstrated in families of subcritical maps [8,12,13]) should be attributed to the fact that in averaging over the Farey level the contribution of rotation numbers with large continuous fraction entries m_i is relatively small. To observe the q^{-3} law one must travel along the mode-locked structure ascending not from level to level (as we do) but from denominator to the next denominator. (For analysis of $\kappa(1)$ see also [14].)

5. Farey separation for different parameter scales

Now we are going to leave the subspace of critical maps and consider the transformation of scales in the whole parameter space – along and across the critical border. The former direction is described by the characteristic scale ω. To characterize the transcritical direction we have earlier introduced three specific distances from the critical line – two supercritical (k_1 – the shortest distance to the line of period-doubling bifurcation and k_2 – the distance to the "chaotic" point Φ – see fig. 1) and one subcritical – the distance h at which the width of the locked region is half of the critical value. The computations for 1023 rationals from 10 Farey levels for the family (1) give evidence that all these three scales are practically equivalent. Statistically they are interconnected by the direct proportions: for most of the studied lockings with non-small denominators we observed $k_2 \approx 2.21 k_1$ and $h \approx 1.15 k_1$ with relatively small individual variations (fig. 5). Mean logarithms of all three scales decrease from level to the next Farey level with an increment tending to (naturally) the common limit 0.402... which is in fact nothing but 2χ of eq. (6).

Having plotted $\log(\omega)$ versus $\log(k_1)$ (or $\log(k_2)$ or $\log(h)$) we observe the already familiar separation over the Farey tree; this can be seen in fig. 6 for the same 255 mode-lockings as in fig. 2. To improve the resolution of fig. 6a, we again employ the skewed coordinates in fig. 6b where the calculated slopes of the linear fits for the layers are given near the corresponding lines and the distances between the layers are shown to the right. The characteristics of the distribution in Farey levels from the third to the twelfth for all three transcritical scales are given in table 3. We see that the distances between the layers tend to the previously calculated value $b = 0.236...$ of eq. (3) whereas the values of slopes eventually become close to $\frac{1}{2}\kappa$. To understand the reason of this coincidence one is to look at the plot of $\log(k_2)$ versus $\log(d)$ (see fig. 7), where d is the characteristic X-scale of the critical locking, and to notice that, notwithstanding the Farey level, the relation $k_2 \cong 29d^2$ holds rather accurately; this imposes evident consequences for k_1 and h. This point also means that 2χ should be the only possible value for the increment of mean logarithm of any transcritical characteristic scale. The explanation of this simple dependence between X-scale and transcritical characteristics will be postponed until the following section[#2].

[#2]As shown by the referee (to whom we are grateful for this remark), the relation $h \propto d^2$ might not hold for the mode-lockings with unbounded entries in the continuous fraction expansion. However, the effects of this kind can be observed only in very high Farey levels; for the rotation numbers with moderate numerators and denominators (which are, surely, the first and most important to be met in applications) the proportionality of transcritical scales to the second power of the X-scale is well-confirmed.

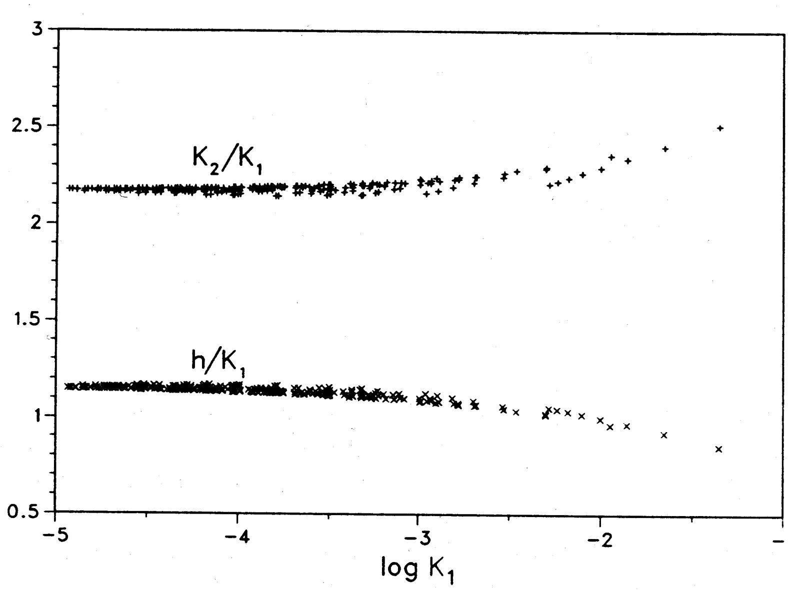

Fig. 5. Ratios k_2/k_1 and h/k_1 of transcritical sizes for 255 mode-lockings in the family (1).

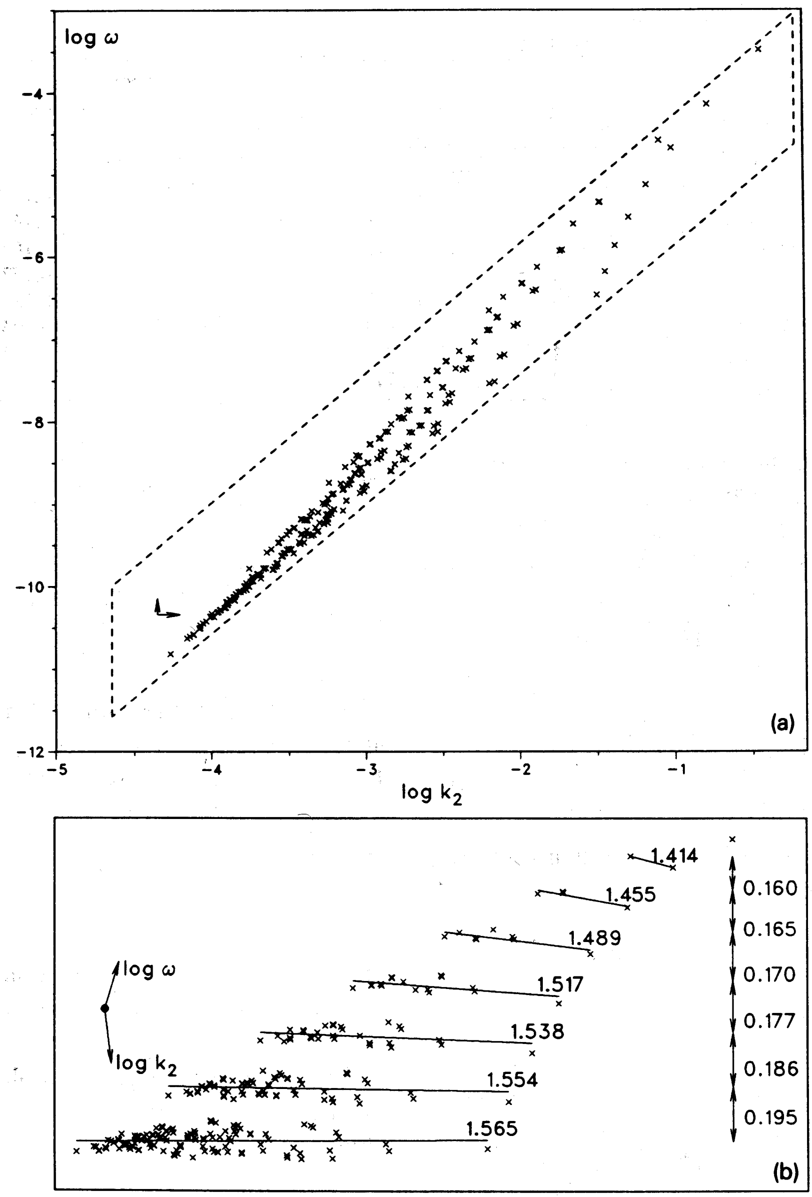

Fig. 6. Dependence of log(ω) on log(k_2) for 255 critical lockings in the family (1).

6. Formulation in terms of rotation function coefficients

The phenomena of Farey separation in the mode-locking structure may be described with the use of another language – in terms of the rotation function $f^q(x)$, which is in fact the most full and relevant characteristic of the p/q-locking. From $f^q(x)$ and its parameter dependence one may gain all the necessary information on various aspects of the mode-locking (and not only the characteristics scales ω, d, etc., which were defined above with some portion of arbitrariness). Again let us take the critical case first

Table 3
Characteristics of layers of ω versus transcritical scales.

Number of the layer	Slopes of layers			Distances between layers		
	for k_1	for k_2	for h	for k_1	for k_2	for h
$j=3$	1.5054	1.4138	1.7671			
$j=4$	1.5178	1.4546	1.7077	0.1981	0.1603	0.3657
$j=5$	1.5315	1.4892	1.6663	0.1976	0.1646	0.3415
$j=6$	1.5437	1.5170	1.6368	0.1996	0.1699	0.3226
$j=7$	1.5536	1.5382	1.6157	0.2036	0.1771	0.3069
$j=8$	1.5612	1.5538	1.6008	0.2087	0.1857	0.2934
$j=9$	1.5667	1.5649	1.5903	0.2140	0.1947	0.2820
$j=10$	1.5706	1.5711	1.5830	0.2190	0.2025	0.2724
$j=11$	1.5733	1.5753	1.5779	0.2233	0.2104	0.2646

and consider the general case of singularity of zth order. On the critical line (parameterized by Ω) inside the parameter interval related to $\rho = p/q$ there is the "superstable" point Ω_0 in which $f^q(0) = p$. (These are the "centroid" points discussed earlier in [8,9]). In the vicinity of this point

$$f^q(x, \mu) = p + F(\mu)\ \mu + G(\mu)\ x|x|^{z-1} + \cdots, \tag{8}$$

with $\mu \equiv \Omega - \Omega_0$. Consider the expansion of $F(\mu)$ and $G(\mu)$ into powers of μ:

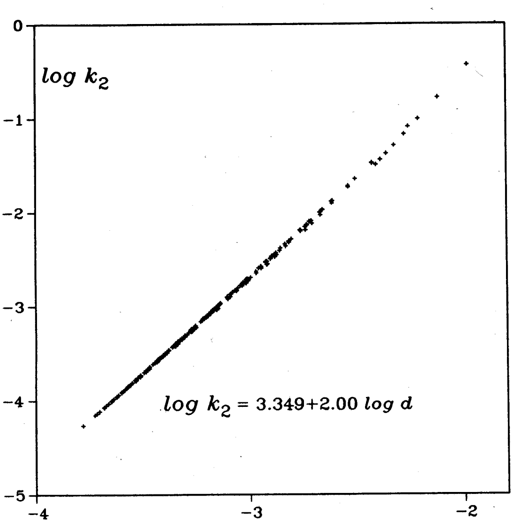

Fig. 7. Dependence of $\log(k_2)$ on $\log(d)$ for 255 critical lockings in the family (1).

$$F(\mu) = \frac{\mathrm{d}f^q(0)}{\mathrm{d}\mu} = F_0 + F'\mu + \tfrac{1}{2}F''\mu + \cdots, \tag{9}$$

$$G(\mu) = \left.\frac{\partial f^q(x)}{\partial(x|x|^{z-1})}\right|_{x=0} = G_0 + G'\mu + \tfrac{1}{2}G''\mu^2 + \cdots. \tag{10}$$

The above mentioned characteristic scales d and ω may be evaluated in terms of coefficients of truncated expansion in the point Ω_0:

$$d \cong (G_0 z)^{1/(1-z)}, \quad \omega \cong 2\,\frac{z-1}{zF_0}\,(G_0 z)^{1/(1-z)} \tag{11}$$

(for the family (3) with cubic singularity the relative error of these estimates is less than 0.07).

To satisfy eq. (4) the coefficients F_0 and G_0 must be interrelated through the values of κ and $B = \exp(b)$:

$$F_0 \simeq B^j G_0^{(\kappa-1)/(z-1)} \tag{12}$$

(as earlier j is the number of the Farey level to which p/q belongs). Using the chain differentiating we calculated F_0 and G_0 for many lockings in various families and found that in plots of $\log(F_0)$ versus $\log(G_0)$ the Farey separation was apparent. For the family (1) with previously computed κ and b the distance between the layers should tend to 0.237 and their slope tend to 1.074. The set of 255 points corresponding to fig. 2 is presented in fig. 8a (again in the skewed coordinate frame with the last layer being horizontal) and we see that the numerical data are in good accordance with the estimates (12).

Within the mode-locked intervals the graphs of $F(\mu)$ and $G(\mu)$ resemble slightly asymmetrical parabolas with the maxima located near $\mu = 0$. Therefore the main corrections to eq. (11) are due to the terms proportional not to the first parameter derivatives F' and G' but to the second ones – F'' and G''. To avoid quantitative changes in eq. (4) these corrections should satisfy the relations

$$F'' \simeq F^{(3\kappa-1)/(\kappa-1)} B^{-2j/(\kappa-1)} , \tag{13}$$

$$G'' \simeq G_0^{1+2\kappa/(z-1)} B^{2j} , \tag{14}$$

which yields the statistical dependence between F'' and G'':

$$G'' \simeq (F'')^{(2\kappa+z-1)/(3\kappa-1)} B^{-j(3z-1)/(3\kappa-1)} . \tag{15}$$

This implies the Farey level separation also for the parameter derivatives of the expansion coefficients of $f^q(x)$. For the family (1) the slope of new layers should tend to 0.983 and the distance between them should tend to 0.224. This conjecture was checked numerically; the pot of $\log(G'')$ versus $\log(F'')$ in the oblique-angled coordinates is presented in fig. 8b and the computed numerical characteristics of the separation match our expectations.

To describe the transcritical case the expansion of $f^q(x)$ should include the linear in x-term and also one more parameter (along with μ),

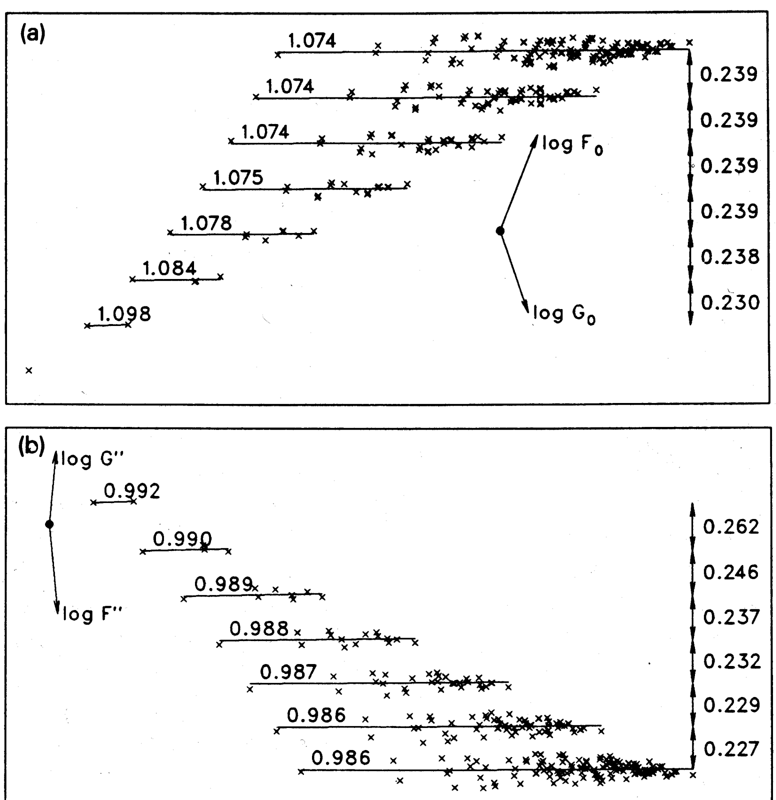

Fig. 8. Farey separation in coefficients of the rotation function: (a) $\log(F_0)$ versus $\log(G_0)$; (b) $\log(G'')$ versus $\log(F'')$.

characterizing the decline from the critical boundary; in the sine map (1) this is $\xi = K - 1$. For the most typical case $z = 3$ we have

$$f^q(x, m, \xi) = p + F(\mu, \xi)\,\mu + P(\mu, \xi)\,\xi + Q(\mu, \xi)\,\xi x + G(\mu, \xi)\,x^3 + \cdots , \tag{16}$$

with

$$P(\mu, \xi) = \frac{\partial f^q(0)}{\partial \xi}, \quad Q(\mu, \xi) = \left.\frac{\partial^2 f^q(x)}{\partial \xi\, \partial x}\right|_{x=0}$$

(the term with $x^2\xi$ may be dropped with proper choice of ξ – this corresponds to choosing ξ "ortogonal" to the critical line). The coefficient Q of the linear term may be expanded as $Q(\mu, \xi) = Q_0 + \mu Q_\mu + \xi Q_\xi + \cdots$. In the shortest truncation of $f^q(x)$ the scales k_1, k_2 and h are expressed solely through Q_0:

$$k_1 \cong \frac{1}{Q_0}, \quad k_2 \cong \frac{3}{Q_0}, \quad h \cong \frac{1 - 4^{-1/3}}{Q_0}. \tag{17}$$

Let us now show that Q_0 is proportional to G_0. For $\mu = 0$ we have the trajectory $x = 0$, $x_1 = f(x), \ldots, f^q(x) = f(x_{q-1})$. Applying chain differentiating and taking into account that $\partial x_m/\partial x = 0$ for $m \geq 1$, we get

$$Q_0 = \frac{\partial^2 f(0)}{\partial x\, \partial \xi} \prod_{i=1}^{q-1} \frac{\partial f(x_i)}{\partial x_i},$$

$$G_0 = \frac{1}{6}\, \frac{\partial^3 f(0)}{\partial x^3} \prod_{i=1}^{q-1} \frac{\partial f(x_i)}{\partial x_i}. \tag{18}$$

One can see that Q_0 is proportional to G_0 which in its turn is the inverse second power of the X-scale d (see the estimate (11)) and the proportionality coefficient for the families like (5), where Ω enters additively, is independent of the rotation number. Though the estimates (17) are rather rough, the general tendency prevails and the transcritical scales are proportional to d^2.

Summarizing the results of the last section we want to stress that the relations between the coefficients of expansion of the function $f^q(x)$ and between their parameter derivatives exhibit statistical scaling properties of the same kind as the characteristic sizes of mode-lockings in the parameter space and on the X-axis. The layer to which the locking belongs is entirely determined by the number of the Farey level of the rotation number. This gives further evidence to our hypothesis that the number of the Farey level is the natural and convenient coordinate for comparison of mode-locking scales in the parameter space and on the X-axis.

The obtained results allow to predict with considerable accuracy the parameters of mode-locked intervals from the high Farey levels provided the properties of few first lockings are calculated. Further activity in this direction should incorporate the described phenomena into the Farey tree renormalization scheme [8,9,11].

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