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LETTER TO THE EDITOR

Escape exponent for transient chaos and chaotic scattering in non-hyperbolic Hamiltonian systems

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Abstract. It is shown that in non-hyperbolic Hamiltonian systems where correlations decay as a power law, transient chaos and chaotic scattering demonstrate different power laws. Numerical evidence of this effect for the stadium billiard is presented.

Problems of transient chaos and of chaotic scattering are very similar [1]. In both cases a chaotic repeller exists in phase space, and trajectories eventually escape from its vicinity. The difference between the two problems is in the initial conditions. In the case of transient chaos one usually assumes that there exists a system with statistically stationary chaotic behaviour (strange attractor for dissipative systems, invariant chaotic set in the Hamiltonian case), and then trajectories are allowed to escape (for example, by changing parameters). In the case of chaotic scattering one usually has a set of particles coming from infinity, and these particles escape the region of irregular behaviour after some time. The difference, thus, is in the initial distribution: for transient chaos particles are initially distributed according to invariant measure on the chaotic set, while for scattering the initial distribution is concentrated on some subsets in the phase space disjoint from the chaotic set. As an example, consider Sinai's billiard system. If we make a hole in the wall of the billiard and consider particles coming through the hole and leaving the billiard after a number of collisions, we have the scattering problem. Transient chaos corresponds to a situation in which initially the particles move inside the closed billiard, and then the hole is opened.

In the case of a hyperbolic repeller the number of non-escaped particles decays exponentially in time both for transient chaos [2, 3] and for chaotic scattering [5]. However, many dynamical systems do not exhibit strong hyperbolic properties. In particular, Hamiltonian systems with divided phase space, where regions of chaotic and regular motions coexist, demonstrate intermittent behaviour. Trajectories may be held up for a very long time near the border with regular regions, where mixing is very weak. This results in power-law tails of the correlation function [6–11]. In these systems we expect the number of non-escaped particles to decay as a power law [12]:

$$N(t) \sim t^{-\alpha}. \qquad (1)$$

Here N is the number of particles which escape after time t. The main characteristics of such non-hyperbolic Hamiltonian systems is the exponent α. The aim of this letter

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is to show that in these systems the exponents for transient chaos α_t and for chaotic scattering α_s are related by $\alpha_t = \alpha_s - 1$.

Qualitatively, the difference between escape indices for the two cases results from the following. Trajectories stay for a long time near the border of chaos. Because the invariant measure for Hamiltonian systems is the Lebesgue measure, the probability for a trajectory inside the chaotic region to come close to the border is very small. Thus, if the initial distribution is inside the chaotic region, the case of chaotic scattering, only a small number of trajectories are held up for a long time. If the initial distribution is non-zero near the boundary of chaos, the case of transient chaos, a larger number of trajectories are held up and the exponent is smaller.

Quantitatively, we derive α using the Chirikov–Shepelyansky model [6]. In this model motion along the border between chaotic and regular regions is assumed to be highly chaotic, while motion along the transverse coordinate x is diffusive. For the probability distribution they proposed the following phenomenological model (see also [10])

$$\frac{\partial W(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(x^{\beta}\frac{\partial W(x,t)}{\partial x}\right) \qquad \beta > 2. \tag{2}$$

The diffusion rate here is proportional to x^{β} and vanishes at the border of chaos where $x = 0$. One can easily see that the homogeneous solution $W = \text{constant}$ satisfies (2). But in the problems of transient chaos and chaotic scattering it is natural to complement (2) with the boundary condition

$$W(L, t) = 0$$

which describes escape of particles at $x = L$. We cannot solve (2) exactly with this boundary condition. Therefore we consider (2) in a semi-infinite domain $0 < x < \infty$ and assume that the particles outside the interval $0 < x < L$ 'effectively escape'. This approximation is good because the diffusion rate is large for large x and the probability for particles to come back from large to small x is very small. Thus the quantity of interest is the total number $N(t)$ of particles in the interval $0 < x < L$:

$$N(t) = \int_0^L W(x,t)\,\mathrm{d}x. \tag{3}$$

We use the known Green function for (2) [10]:

$$G(x,t,\xi) = (\beta-2)(x\xi)^{-1/2} y\eta\, \mathrm{e}^{-y^2-\eta^2} I_p(2\eta y) \tag{4}$$

which satisfies the initial condition $G(x, 0, \xi) = \delta(x - \xi)$. Here

$$y = (\beta-2)^{-1} t^{-1/2} x^{-(\beta-2)/2} \qquad \eta = (\beta-2)^{-1} t^{-1/2} \xi^{-(\beta-2)/2}$$

$p = (\beta-1)/(\beta-2)$ and I_p is the modified Bessel function.

For the problem of chaotic scattering we are interested in an initial distribution outside the border of chaos. Therefore it is sufficient to consider a localized initial distribution of the form $W(x, 0) = \delta(x - \xi)$, in which case the solution $W(x, t)$ is given by (4). Substituting (4) in (3) and changing the variable of integration from x to y, we obtain

$$N(t) \sim t^{-(\beta-1)/2(\beta-2)} \int_{c_1 t^{-1/2}}^{\infty} y^{-1/(\beta-2)}\, \mathrm{e}^{-y^2-\eta^2} I_p(2\eta y)\,\mathrm{d}y. \tag{5}$$

Because $\eta \sim t^{-1/2} \to 0$ for $t \to \infty$, we can approximate $I_p(2\eta y)$ for small ηy as $I_p(2\eta y) \approx (\eta y)^p$. Substituting this in (5) we obtain

$$N(t) \sim t^{-(\beta-1)/(\beta-2)} \tag{6}$$

which yields $\alpha_s = (\beta-1)/(\beta-2)$ for the exponent of chaotic scattering.

For transient chaos we choose a uniform initial distribution in the interval $0 < x < L$: $W(x, 0) = W_0$. Then

$$W(x, t) = W_0 \int_0^L G(x, t, \xi)\, \mathrm{d}\xi$$

and (3) is reduced to

$$N(t) = W_0 \int_0^L \mathrm{d}x \int_0^L \mathrm{d}\xi\, G(x, t, \xi).$$

Changing the variables of integration to η and y, we have

$$N(t) \sim t^{-1/(\beta-2)} \int_{c_1 t^{-1/2}}^{\infty} \mathrm{d}y \int_{c_1 t^{-1/2}}^{\infty} \mathrm{d}\eta\, \mathrm{e}^{-y^2-\eta^2} (y\eta)^{-1/(\beta-2)} I_p(2\eta y).$$

One can easily check using the above relation for $I_p(z)$ for small z, and also the asymptotic $I_p(z) \sim \mathrm{e}^z (2\pi z)^{-1/2}$ as $z \to \infty$, that the double integral converges at both limits yielding

$$N(t) \sim t^{-1/(\beta-2)} = t^{-(\beta-1)/(\beta-2)+1}. \tag{7}$$

Comparing (6) and (7) we obtain that the exponent for transient chaos $\alpha_t = 1/(\beta-2) = aaa\alpha_s - 1$.

The exponents can also be easily estimated for a one-dimensional model of non-hyperbolic Hamiltonian systems [13, 14]. The model is a symmetrical one-dimensional mapping $z \mapsto f(z), f(-z) = -f(z)$ with marginally linearly stable fixed points $z = \pm 1$: $f(-1+z) \approx -1 + z + \text{constant } z^\rho$ for $0 < z \ll 1$. In the numerical simulations presented below we use the following implicit expression for $f(z)$ [14]:

$$\begin{aligned} z &= \frac{1}{2\rho}[1+f(z)]^\rho & & 0 < z < (2\rho)^{-1} \\ z &= f(z) + \frac{1}{2\rho}[1-f(z)]^\rho & & (2\rho)^{-1} < z < 1 \\ f(-z) &= -f(z). \end{aligned} \tag{8}$$

The advantage of the model (8) is that it has a uniform invariant probability density and thus mimics Hamiltonian systems.

The problems of scattering and transient chaos may be posed for the system (8) if we make a 'hole' $0 < z_1 < z < z_2 < 1$ where trajectories escape. Let us estimate the exponents for this system. In the problem of transient chaos we start from the uniform distribution. The main contribution to $N(t)$ is trajectories beginning near the marginally stable fixed points. It is easy to show that a trajectory that starts at distance ξ from the fixed point leaves its vicinity after time $\sim \xi^{1-\rho}$ and thus

$$N(t) \sim t^{1/(1-\rho)}. \tag{9}$$

In the scattering problem the initial distribution is concentrated inside the interval $z_1 < z < z_2$. Here the main contribution to $N(t)$ is from the trajectories that fall in the vicinity of the origin and are mapped to the vicinities of the fixed points. The probability to fall in the interval $[-1, -1+\xi]$ is proportional to ξ^ρ and hence

$$N(t) \sim t^{\rho/(1-\rho)} = t^{1/(1-\rho)-1} \tag{10}$$

Again we find that the difference between the exponents for transient chaos and chaotic scattering is equal to 1.

We now present numerical evidence for the formula derived in (9) and (10). In figure 1 we check (9), (10) for the one-dimensional model (8). In figure 2 we present

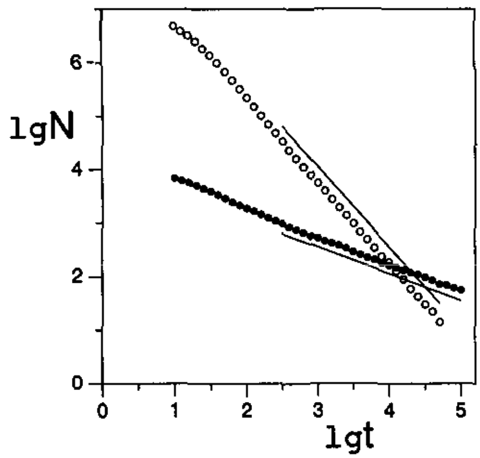

Figure 1. N versus t for scattering (open circles) and transients (filled circles) in the mapping (8) with $\rho = 3$. Lines have slopes 1.5 and 0.5 in accordance with (9), (10). The logarithms here and in figure 2 are decimal.

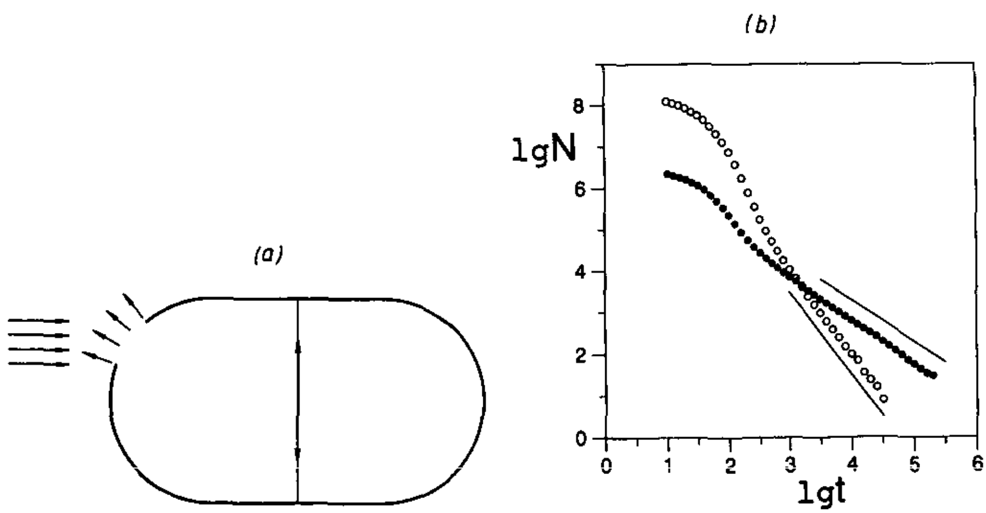

Figure 2. (a) Configuration of the stadium billiard system. (b) N versus t for scattering (open circles) and transients (filled circles) in the stadium billiard. Lines have slopes 1 and 2.

data for transient chaos and chaotic scattering for the Bunimovich stadium billiard [15]. It is known that in this billiard correlations decay as a power law [15, 16]. The role of the 'border of chaos' here is played by weakly unstable periodic trajectories perpendicular to the parallel walls (see figure 2(a)). We make a hole in the billiard's wall. In the scattering problem, particles are injected through the hole and the number of collisions before their escape was computed. In the transient chaos problem particles were initially distributed according to the known invariant density [15]. The exponents obtained are in agreement with (6), (7).

In conclusion, we have shown that in non-hyperbolic Hamiltonian systems exponents for the problems of transient chaos and chaotic scattering differ by one. This is caused by intermittent behaviour of trajectories which 'stick' near the boundaries of chaotic regions.

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