DOES DIMENSION GROW IN FLOW SYSTEMS?

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ABSTRACT. Flow systems are determined as those with linear and nonlinear convective instability. It is shown, that dimension in such systems does not grow with the distance along the flow. Transition to turbulence occurs through filling of the phase space by a low-dimensional object.

Chaotic behavior of spatially extended dynamical systems is intensively investigated now. It supposed that this study may give insight on the problem of turbulence. In hydrodynamical turbulence mean flow is usually presented (for example in jets and wakes). From the theoretical point of view flow systems are specified by a special type of linear instability - *convective instability* (not to be mixed with thermal convection). For absolute instability a localized initial disturbance grows at the same space region where it was initially imposed. Thus the formulation of problem for media with absolute instability (e.g., Raleigh-Benard convection, Taylor vortices) is the same as for ordinary differential equations: starting from "arbitrary" initial conditions one follows the time evolution of the field and determines, whether this evolution is chaotic or not. For *convective* instability a localized initial disturbance simultaneously grows and moves away. So if we interested in the field behavior in a finite spatial region, it is not sufficient to impose initial conditions. A nontrivial state may be observed only if there are constantly driving perturbations. In an experiment these perturbations may be natural (usually random) or artificial (with any dynamical behavior). Thus, for convectively unstable systems the problem is of nonlinear dynamical transformation of an external signal. If the region of external driving is located near the end of the system, then one can say about spatial amplification and development of perturbations. Such a problem was considered for noisy [1] and regular [2] boundary perturbations. Recently there were also attempts to apply dynamical concepts of turbulence to experiments [3,4,5]. In particular, in refs. [4,5] the attempts were made to describe quantitatively the spatial development of turbulence by measuring spatial growth of dimension. The purpose of the present paper is to show that dimension does not grow in flow systems. Transition to turbulence occurs not through dimension growth, but through filling of the phase space by a low-dimensional object.

Let us define first what we mean by a flow system. A system will be called "flow" if it is convectively unstable both in linear and nonlinear regimes. For linear systems the

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criteria for convective and absolute instability are well-known (see e.g., [6]). For secondary instabilities of the nonlinear state one may use so–called velocity dependent Lyapunov exponents, defined in ref. [7]. Convective nonlinear instability occurs when these Lyapunov exponents are positive only for strictly positive (negative) velocities, and are negative for zero velocity of a reference frame.

From the consideration of Lyapunov exponents it easy to conclude that dimension cannot grow in flow systems. Indeed, it is known that dimension af a dynamical system is limited by above by the Lyapunov dimension D_λ, defined by the Kaplan - Yorke formula [8,9]:

$$D_\lambda = M + \frac{\lambda_1 + \cdots \lambda_M}{|\lambda_{M+1}|}, \qquad \text{where} \qquad \lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots$$

are Lyapunov exponents and the integer part M is defined by

$$\sum_1^M \lambda_i \geq 0 > \sum_1^{M+1} \lambda_i$$

For flow systems, as they were defined above, all Lyapunov exponents are negative. However, we must take into account the Lyapunov exponents of the external signal, which is transformed by the flow system. It is natural to connect with a periodic signal one zero Lyapunov exponent, with a two–frequency quasiperiodic signal – two zero exponents. For a chaotic signal it is natural to take into account as many Lyapunov exponents, as are necessary for determination its Lyapunov dimension. Combining the Lyapunov exponents from the external signal and from the flow system, we can immediately conclude that there can be no chaotisation of periodic and quasiperiodic signals. Nontrivial complication of the process occurs only for quasiperiodic signal [2,10,11,12], where torus is folded and stretched in a phase space. The dimension of the torus does not grow, but this torus appears to fill the phase space. We'd like to illustrate this with calculation of signal evolution along a convectively unstable coupled map lattice [13]

$$u_{n+1}(x) = \varepsilon f(u_n(x-1)) + (1-\varepsilon) f(u_n(x))$$

where $f(u) = 1 - 2|u|$ is nonlinear function, n and x – discrete time and space. Convective instability occurs for $\varepsilon > 0.5$. At the fig.1 we present $u_n - u_{n+1}$ plots for input quasiperiodic signal and for different spatial coordinate x.

More complex is development of a chaotic external signal. Suppose that this signal has Lyapunov exponents $\gamma_1 > 0$, $\gamma_2 = 0$, $\gamma_3 < -\gamma_1$ and its Lyapunov dimension is therefore $D_{\lambda\,\mathrm{inp}} = 2 + \gamma_1/|\gamma_3|$. If the inequality $\lambda_1 > \gamma_3 \geq \lambda_2 \ldots$ is satisfied, then the output Lyapunov dimension of the signal, transformed by the convectively unstable system, is

$$D_{\lambda\,\mathrm{out}} = \begin{cases} 2 + \gamma_1/|\lambda_1| & \text{for} \quad \lambda_1 + \gamma_1 < 0 \\ 3 + (\gamma_1 + \lambda_1)/|\gamma_3| & \text{for} \quad \lambda_1 + \gamma_1 \geq 0 \end{cases}$$

Thus, the dimension can grow a little, but this growth may be considered as a result of linear filtering of the input signal [14]. Nevertheless, we can not get significant growth of dimension unless we are not close to the point of transition from convective to absolute instability (where Lyapunov exponents of the flow system are close to zero). Spatial development of turbulence here also occurs via filling of the phase space by a low-dimensional strange attractor.

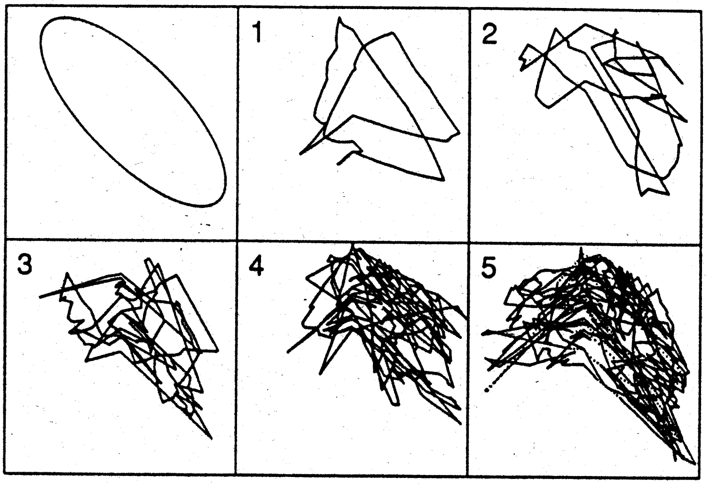

Figure 1. Spatial development of quasiperiodic signal in the convective coupled map model; $\varepsilon = 0.7$, $u_n(0) = 0.1 + 0.8\sin(2\pi\rho n)$, $\rho = (\sqrt{5} - 1)/2$.

In conclusion we'd like to discuss, why in processing of real data in [4] a visible growth of dimension along the flow was obtained. Usually one obtains an estimation of the dimension via Grassberger–Procaccia [15] algorithm. In this procedure one computes a number of neighbors of a given point in a phase space. When the phase space is filled by a low-dimensional object, as it is the case in flow systems, a smallest scale cutoff, below which this algorithm gives true dimension, decreases with flow distance, probably, exponentially. So for valid dimension estimation one has to increase proportionally the number of points in the data record, what is practically impossible.

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