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Manifestations of classical and quantum chaos in nonlinear wave propagation

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We consider the evolution of waves governed by a periodically perturbed nonlinear Schrödinger equation. The system is a nonlinear generalization of the quantum kicked rotator. We study the soliton chaotic motion, its destruction and reversibility properties. It is found that nonlinearity does not destroy the quantum suppression of chaotic diffusion.

At present, the properties of dynamical chaotic motion in classical Hamiltonian systems with few degrees of freedom are reasonably well understood. Their most important characteristic is exponential local instability of motion. This instability results in a practical irreversibility of dynamics as well as in a mixing in phase space and correlations decay. A well-known simple example of such chaotic behavior is given by the standard map [1].

During the last decade much attention was devoted to the manifestation of these classical features in the corresponding quantum systems [2,3] or in classical linear wave systems [4,5]. It was shown that the quantum motion is stable and reversible [6] and that classical local instability manifests itself only during a short-time interval as a rapid destruction of narrow quasiclassical wave packets [7,8]. One of the most important and interesting quantum-mechanical effects is the suppression of the classically chaotic diffusive excitation.

The basic model used to understand this behavior is the quantum version of the standard map: the so-called "kicked rotator," namely a rotator under a time-periodic, δ-kick perturbation. Indeed it is known that when the perturbation strength exceeds a critical value called the chaos border, the classical motion becomes diffusive and the energy of the rotator increases, in the average, linearly with time. Instead, in the quantum case, one observes the localization phenomenon: the excitation settles after a while to a steady-state distribution which decays exponentially moving away from the initially excited unperturbed state [2,6,8,9]. As a consequence, unlike the classical, the quantum rotator energy is limited by some finite value. It is interesting to remark that the quantum kicked rotator describes as well the motion of classical waves such as the propagation of light in waveguides [4,5]. Highly nontrivial problems arise if these waves propagate in a nonlinear medium and are governed, for example, by the nonlinear Schrödinger equation. It is, therefore, extremely interesting to understand how nonlinearity modifies the general picture drawn from the kicked rotator model. In particular one would like to know to what extent the main features of quantum chaos, i.e., destruction of wave packets, stability and reversibility of motion, quantum suppression of diffusion, and localization, etc. survive to the introduction of nonlinearity.

In this paper we study the nonlinear Schrödinger equation under time-periodic δ-kick perturbation described by the following equation:

$$i\frac{\partial\psi}{\partial t}=-\frac{\partial^2\psi}{\partial x^2}-\beta|\psi|^2\psi+k(\cos x)\psi\sum_{m=-\infty}^{+\infty}\delta(t-mT)\,, \tag{1}$$

where β and k are two parameters which measure the nonlinearity and the kick strength, respectively. Here and in the following we set $\hbar=1$. The probability $|\psi|^2$, being an integral of motion, is normalized to one. We consider the motion on a ring, so that $\psi(x,t)=\psi(x+2\pi,t)$.

The system (1) can be considered as a model for propagation of nonlinear waves in a medium as in optical fibers: the last term may be interpreted as a change of the optical density inside the waveguide. Also the same model (1) approximately describes the propagation in waveguides with longitudinal sinusoidal modulation of the boundary. For these cases time in Eq. (1) plays the role of the longitudinal direction along which waves propagate.

In the case $k=0$ Eq. (1) reduces to the usual completely integrable nonlinear Schrödinger equation possessing well-known soliton solutions [10,11]. In the other limiting case, corresponding to $\beta=0$ Eq. (1) reduces to the quantum kicked rotator model [2,8]. In the classical linear limit ($\beta=0$, $k\to\infty$, $T\to 0$, and $kT=$const) the motion is described by the standard map. Another connection with classical mechanics can be obtained by considering solitons as classical particles [12,13]. As shown below the motion of solitons is governed by the same standard map; the question arises however for how long a time the soliton solution will be valid. In the unperturbed case ($k=0$) the one-soliton solution is given by

$$\psi(x,t)=\frac{1}{2}\left(\frac{\beta}{2}\right)^{1/2}\frac{\exp\left[i\frac{v}{2}\left(x-x_0-\frac{v}{2}t\right)+i\frac{\beta^2}{16}t\right]}{\cosh\left[\frac{\beta}{4}(x-x_0-vt)\right]}\,,$$
$$\int_0^{2\pi}|\psi|^2dx=1\,, \tag{2}$$

where v is the velocity of the soliton and x_0 its initial center position. The characteristic width of the soliton in coordinate space $\Delta x = 4/\beta$ is assumed to be much less than 2π, while the width in Fourier space is $\Delta n \approx \beta/4$. Notice that the maximum in the Fourier components (in the unperturbed levels for the quantum kicked rotator) for such a solution is located at $v/2$.

The effect of the kick can be represented by multiplication of the ψ function:

$$\psi \rightarrow \bar{\psi} = e^{-ik\cos x}\psi ; \tag{3}$$

therefore for large β, when the size of the soliton is small enough, its center position and velocity can be described by the standard map:

$$\bar{v} = v + 2k\sin x , \quad \bar{x} = x + \bar{v}T . \tag{4}$$

The map (4) is obtained by inserting Eq. (2) in Eq. (3) and expanding $\cos x$ near the center of the soliton. The classical chaos parameter is $K = 2kT$ and will determine here whether the soliton motion is stable or chaotic.

In order to check that the map (4) is a good description for the soliton motion, we numerically integrated Eq. (1) and computed the average soliton position $\langle x \rangle_m$ after the mth kick, as well as the soliton velocity determined by $(\langle x_{m+1} \rangle - \langle x_m \rangle)/T$. Figure 1 shows a plot of two typical soliton trajectories in the phase space, one in the stable and the other in the chaotic region. Classical phase-space points are here substituted by segments centered at the soliton position; the length of the segment is equal to the soliton width Δx. From the computed average positions of the above chaotic trajectory we calculated the function $g(\langle x \rangle_m) = (\langle x \rangle_{m+1} - 2\langle x \rangle_m + \langle x \rangle_{m-1})/(2kT)$, and compared it with the curve $g(x) = \sin x$. The comparison between these two functions shown in Fig. 2 indicates that the standard map gives a good description of the soliton's motion.

However the approximation involved in considering the soliton as a point particle is valid only up to some finite time after which the effects of finite width must be taken into account. Indeed, as it is seen from Eq. (3), after the action of a kick the soliton shape is slightly disturbed due to second and higher-order terms in the expansion of $\cos x$ near the center of the soliton. The effect of one kick is a decrease of soliton amplitude by a quantity proportional to k^2/β^3 [14] and a corresponding increase of nonsoliton wave field. In the approximation in which the radiated wave field is neglected, the soliton lifetime t_s is expected to be proportional to $t_s \approx \beta^4/k^2$ [14]. Numerical results obtained for chaotic soliton motion with parameter $K = 3$ show that the soliton lifetime t_s, defined as the time after which the width of the soliton becomes two times larger than the initial value, can be approximated as $t_s = \alpha\beta^2/k^2$. Numerical data obtained in a wide range of parameters $10 \leq \beta \leq 40$ and $0.2 \leq k \leq 3$ lead to $\alpha \approx 0.33$. The difference from the analytical estimate given above is probably caused by the influence of radiated field on the soliton evolution.

One of the characteristic features of quantum dynamics of classically chaotic systems is the stability property. Indeed, as it is known, classically deterministic chaotic systems are exponentially unstable with respect to the initial conditions. Therefore they are practically irreversible, in spite of the exact reversibility of the equations of motion: this is caused by the exponential growth of errors and finite precision of the computation. On the contrary, numerical computations performed on quantum systems

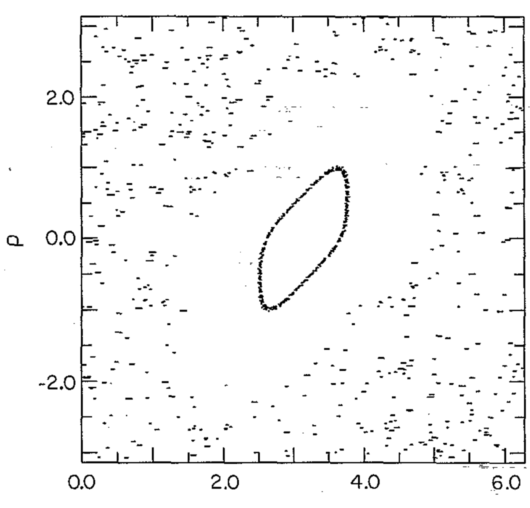

FIG. 1. Two phase-space trajectories with parameters $\beta = 25$, $k = 0.5$, and $T = 2$ (classical K is 2), obtained by numerical integration of Eq. (1) up to 500 periods of the external perturbation. The chaotic and regular trajectories start from an initial soliton condition with $v = 0.1$, $x_0 = 0.1$ and $v = 0$, $x_0 = 2.6$, respectively. Segments are centered around the soliton center position $\langle x \rangle$ and momentum $p = vT$, and have a length equal to the soliton width. While the motion is integrated over an unbounded cylindrical phase space, the momentum is plotted modulus 2π for the sake of clarity.

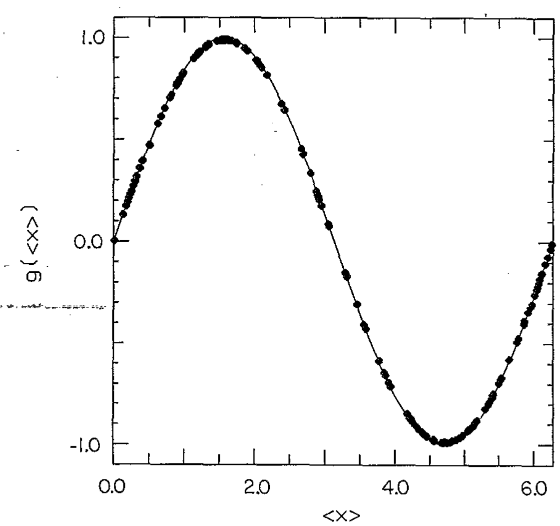

FIG. 2. Comparison between the function $g(\langle x \rangle_m) = (\langle x \rangle_{m+1} - 2\langle x \rangle_m + \langle x \rangle_{m-1})/(2kT)$, computed for the chaotic trajectory of Fig. 1 (dots), and the kick function $g = \sin x$ (solid line).

have shown remarkable stability of the quantum motion: by reversing the time it is possible to recover the initial state with high accuracy [6,15]. For the linear wave propagation this reversibility is used in the phase conjugation technique. The question is whether or not this property is shared by the nonlinear Eq. (1) with nonvanishing β. Notice that Eq. (1) is exactly reversible: if we change the sign of the velocity, namely substitute ψ with ψ^*, halfway between two subsequent kicks, then the wave distribution must return back to the initial state. For initial conditions corresponding to stable region of map (4) it is expected that the motion is reversible, since computer errors grow here with some power of time. Instead, in the chaotic region errors grow exponentially and the standard map (4) is practically irreversible. However, the nonlinear Schrödinger equation is completely integrable (namely, between kicks the motion is regular) and therefore it is not obvious whether or not the motion described by Eq. (1) is really exponentially unstable.

We investigated the instability properties of Eq. (1) by time reversing the equations of motion. The results are shown in Fig. 3 where we plot the probability amplitude in coordinate space for two different soliton initial conditions, one in the stable region and the other in the chaotic region. However, the two solitons have the same initial coordinate so that the two initial probabilities are superimposed in Fig. 3. After 150 periods of the external perturbation the solitons velocity has been reversed and the numerical integration continued for an additional 150 periods. In the stable region the time-reversed soliton recovers its initial position and shape (with the accuracy 10^{-9}). So this motion is fully stable. Instead the soliton moving in the chaotic region does not come back to the initial state. Indeed we determined the exponent of local instability of motion by measuring the difference in the soliton's positions at instants of time symmetric with respect to the time of reversal. The exponent of local instability or KS entropy computed in such a way is $h \simeq 0.6$ and coincides with that of the standard map, while in the stable region $h=0$ and the instability in time follows a power law with power three. Therefore, unlike the case of quantum mechanics of classically chaotic systems, the motion described by Eq. (1) due to the presence of a nonlinear term is really unstable and irreversible. We checked that the instability exists also for initial distributions not of the soliton type. For example, we found irreversible behavior for a homogeneous initial distribution and for the same parameters value of Fig. 3. This means that irreversibility and local instability exists even when the soliton is destroyed. Thus, the phase conjugation for waves propagating in a nonlinear medium may not lead to wave field reversibility. For the above discussion it is essential to make sure that the irreversible behavior we have found is not merely due to the instability of the numerical integration scheme: the evolution in Eq. (1) between kicks was approximated by a sequence of small steps. The step evolution operator was equal to

$$U=\exp\left(-i\tau\frac{\partial^2}{\partial x^2}\right)\exp(-i\tau\beta|\psi|^2)$$

with $\tau=T/M$ and number of small steps M varied from 2000 to 32000 in different runs. The operator U was performed by means of fast Fourier transform with 128–512 components. Our numerical integration scheme exactly conserves the total probability. We also checked that our results were not changed by the increase in the number of

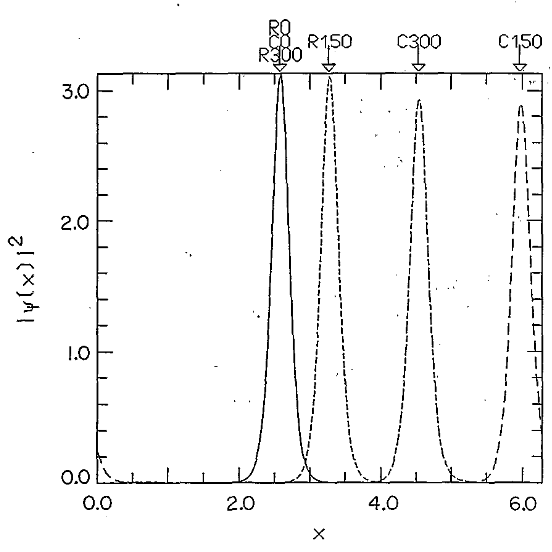

FIG. 3. Check of reversibility in the regular and chaotic regimes. Here $\beta=25$, $k=0.5$, $T=2$, and $x_0=2.6$ for both initial soliton conditions. The initial soliton velocities are taken in regular ($v=0$) and chaotic ($v=1.4$) regions, respectively. The time evolution is reversed at 150 external perturbation periods. The curves labeled R0, C0, and R300 show the two initial regular (R0) and chaotic (C0) solitons and the regular soliton (R300) after complete time reversal ($m=300$) (superimposed); the curve labeled R150 shows the regular soliton at time reversal ($m=150$); the curve labeled C150 the chaotic soliton at time reversal; and the curve labeled C300 the chaotic soliton after complete time reversal.

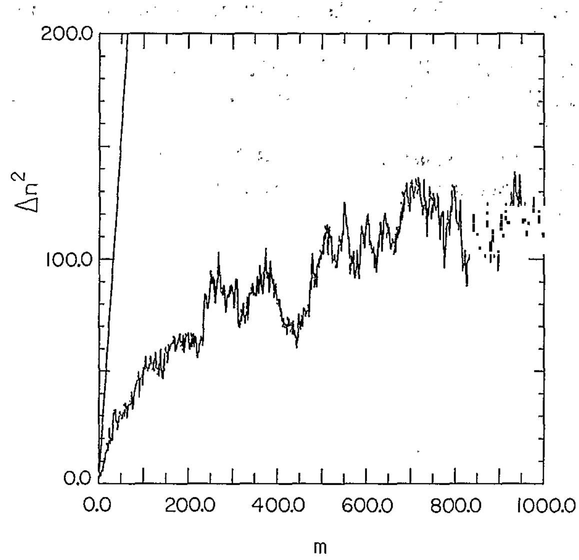

FIG. 4. Plot of the wave packet width in Fourier space Δn^2 vs number of periods m. Here $\beta=10$, $k=2.5$, $T=1$ and classical $K=5$; the initial soliton position and velocity are $x_0=0.2$ and $v=0.2$. The straight line shows the classical diffusion rate $D_{cl}=k^2/2$.

small steps M and in the number of Fourier components.

The other interesting question is whether quantum ($\beta=0$) suppression of diffusive energy growth in the classically chaotic region survives to nonlinearity. The numerically observed behavior reveals that the energy E and the width of the wave packet in Fourier space Δn^2 (these quantities are found to have very similar behavior) look like those in the quantum case. In Fig. 4 we plot Δn^2 versus time for the case $\beta=10$, $k=2.5$, and $K=5$. It is seen that the motion, qualitatively, is analogous to the quantum kicked rotator motion (Refs. [2,6,8]): for some time scale it follows the classical-like diffusive behavior, then it deviates and the diffusion process is suppressed. Notice, that for these parameters values the lifetime of soliton is very small ($t_s \approx 2$). The energy reached at the end of the integration time increases with β (which ranged from 0 to 10) and k (which ranged from 1.25 to 5).

The complexity of the numerical procedure does not allow us to push the integration time up to very long times. At present it is, therefore, not possible to make more definite statements concerning the asymptotic behavior; however, the suppression of classical diffusion is evident. Also our numerical computations indicate that the probability distribution over the Fourier components is roughly exponential even if the stationary state is not yet reached during the interaction time. For example, the amplitude of the highest Fourier component in the case of Fig. 4 was of the order of 10^{-10} after a thousand kicks. The possible physical explanation for diffusion suppression in the nonlinear case is that after soliton destruction the influence of nonlinearity decreases and it does not prevent the interference effects which are responsible for suppression of diffusion in the quantum case.

We integrated Eq. (1) also for the case $K<1$ and found that there is no significant energy growth even for initial conditions not solitonlike or when the lifetime of the soliton is very short. Therefore it appears that the condition for stability of the motion $K<1$ which is known to be valid for the standard map, works also for the nonlinear model (1). In addition, as it is to be expected, in this latter case the motion is always reversible.

In conclusion, we have shown that the kicked nonlinear Schrödinger equation (NSE) model contains properties both of the classical standard map and of the quantum kicked rotator. The most striking result is that nonlinearity does not destroy the suppression of chaotic diffusion. When applied to the propagation of light beams in a perturbed nonlinear waveguide this implies that the beam angular aperture does not grow significantly even for very long waveguides.

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