Chaotic wavefront propagation in coupled map lattices

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Received 2 January 1991; revised manuscript received 9 April 1991; accepted for publication 12 April 1991
Communicated by D.D. Holm

Chaotic wavefront propagation is numerically studied, with the use of coupled map lattices. If the front propagates with "velocity of light", noise-induced period doublings are observed. For the "sublight" propagation there exist resonances leading to a regular temporal wavefront structure.

The problem of turbulence considered as spatial-temporal chaos in distributed dynamical systems has been intensively investigated recently. Such properties as spatial-temporal intermittency [1–3], spatial development of chaos [4,5], statistical properties [6], scaling of spatial bifurcation structures [7], etc. were considered numerically and theoretically. In some situations dynamical spatial-temporal turbulence may be observed experimentally [8]. Many features of spatial-temporal chaos may be modeled with the simplest model – the coupled map lattice (CML), which is discrete both in space and in time [9].

In this paper we investigate the excitation wavefront propagation. This problem is specific for unbounded distributed systems. For nonlinear partial differential equations the speed of front propagation into the linearly unstable state was obtained in refs. [10,11]. Usually, the speed is equal to the speed of a reference frame, for which a convective instability turns into an absolute one [12] [#1]. In the systems with regular behavior the front is a moving region of transition from one regular state to another one (usually from one fixed point to another). Such a front was observed in experiments with Raleigh–Bernard convection [14]. In the chaotic systems the front is a moving region of transition from order to chaos (or from one type of chaos to another – this case will not be considered here). Such a region appears if in an unbounded unstable medium a localized disturbance is initially imposed. The first observation of such a front, as far as I know, was made in ref. [15], where the complex Ginzburg–Landau equation was solved numerically. A localized disturbance gave rise to two fronts: one with higher speed of the type "stationary state–periodic wave", another with less speed of the type "periodic wave–chaos". The front of the type "stable stationary state–chaos" was later observed in a nonlinear mapping with diffusion [3], a system which is discrete in time and continuous in space, and in a CML [16]. This front plays an important role in organization of spatial-temporal intermittency [2,3,16], because it repeatedly appears as a boundary between laminar and turbulent states.

In this paper we describe the front patterns of the "unstable stationary state–chaos" type arising in the CML model. We will show that noise plays an important role in the front structure formation. The spatial discreteness of CMLs is also important and leads to resonances with coherent wavefront behavior.

A coupled map lattice describes the behavior of a field $u_n(i)$, depending on the discrete spatial coordinate $i = ..., -1, 0, 1, 2, ...$ and on the discrete time $n = 0, 1, 2, 3, ...$. The dynamics of this field is gov-

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[#1] For disturbance propagation into a chaotic state this corresponds to a velocity with zero co-moving Lyapunov exponent [13].

erned by a local nonlinear function $f(u)$ and by a spatial linear operator,

$$u_{n+1}(i)=(1-\epsilon)f(u_n(i))$$
$$+\tfrac{1}{2}\epsilon[f(u_n(i-1))+f(u_n(i+1))] . \quad (1)$$

Here the parameter ϵ corresponds to the diffusion constant. The nonlinear function $f(u)$ will be taken in the form of the logistic map with fully developed chaos: $f(u)=4u(1-u)$. The linear spatial operator has no effect on a spatially homogeneous state, so the unstable fixed point $u=0$ of the logistic map gives also a steady ("ground") state of the CML equation (1). But any local disturbance grows and spreads, forming a front on either side. The front speed versus diffusion constant ϵ is presented in fig. 1. For $\epsilon>0.5$ the speed S is equal to 1 which is the maximal possible speed in the CML of the type (1). This critical parameter value may be easily found as follows. A stationary front moving with velocity 1 corresponds to a solution of (1) satisfying $u_n^*(i)=u^*(i-n)$. Thus for stationary front we obtain from (1)

$$u^*(i-1)[(1-2\epsilon)+2\epsilon u^*(i-1)]$$
$$=(1-\epsilon)4u^*(i)[1-u^*(i)]$$
$$+2\epsilon u^*(i+1)[1-u^*(i+1)] . \quad (2)$$

For $u^*(i)=u^*(i+1)=0$ one can easy see that a positive nontrivial solution, $u^*(i-1)=1-1/2\epsilon$, exists only if $\epsilon>\frac{1}{2}$. Eq. (2) defines a two-dimensional mapping

$$u^*(i-1)=F(u^*(i),u^*(i+1)) . \quad (3)$$

The iterations of (3) beginning from the point (0, 0) give the stationary front form. Numerical calculations show that for $i\to-\infty$ iterations of (3) converge to the fixed point $u^*=0.75$, which is stable for the mapping (3).

The calculations with the full nonstationary CML equation (1) show that the leading part of the front follows this stationary solution, while the back part oscillates with period 2, later with period 4, etc. (fig. 2). These spatial period doublings resemble those obtained in ref. [4] for a CML with unidirectional coupling. Indeed, in the reference frame moving with velocity 1 the CML equation (1) takes the form

$$\tilde{u}_{n+1}(i)=\tfrac{1}{2}\epsilon f(\tilde{u}_n(i))+(1-\epsilon)f(\tilde{u}_n(i+1))$$
$$+\tfrac{1}{2}\epsilon f(\tilde{u}_n(i+2)) \quad (4)$$

analogous to that of the uni-directional coupled CML. As in ref. [4], these period doublings are caused by filtering of fluctuations. For small disturbances of a homogeneous state $\tilde{u}(i)=0.75+\tilde{v}(i)$ we obtain in a linear approximation

$$\tilde{v}_{n+1}(i)=-\epsilon\tilde{v}_n(i)-2(1-\epsilon)_n(i+1)$$
$$-\epsilon\tilde{v}_n(i+2) . \quad (5)$$

One can see from (5) that initial disturbances decrease in time. However, for constantly acting disturbances one must consider the solutions of eq. (5)

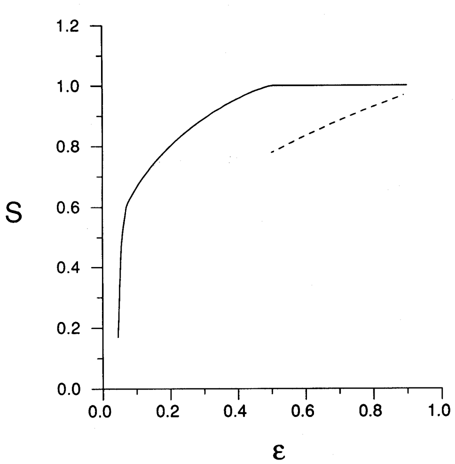

Fig. 1. Wavefront speed versus diffusion constant. (——) Propagation into the "ground" state $u=0$; (---) propagation into the state $u=\frac{3}{4}$.

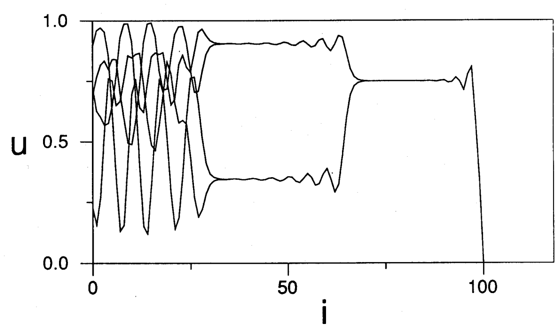

Fig. 2. Wavefront structure for $\epsilon=0.75$. (All the waves in figs. 2–5 are presented in a reference frame moving with the speed of the wavefront.)

in the form $\tilde{v}_n(i) \sim \kappa^{-i} \exp(i\omega n)$. The spatial amplification rate κ happens to be greater than 1, and takes its maximal value for $\omega = \pi$, i.e. for disturbances with temporal period 2. In numerical modeling of CMLs disturbances are caused mainly by truncation errors and are very small, so the disturbances with temporal periods not equal to 2 are either filtered out or dominated by those with period 2, and one observes spatial period doublings.

Thus, in a noisy system [#2] for $\epsilon > \frac{1}{2}$ one observes a front of spatial period doublings, moving with velocity $S=1$, fig. 2 (compare ref. [5]). Noise may be eliminated if one considers the state $u=0.75$ as unexcited and tries to find a front on its base. The speed of this front is presented in fig. 1 by the dashed line, it is always less than 1. Thus, in a noiseless system for $\epsilon > \frac{1}{2}$ two fronts are formed: the first, having velocity 1, of "fixed point–fixed point" type, and the second one, having velocity less than 1, of "fixed point–chaos" type (fig. 3).

For $\epsilon < \frac{1}{2}$ the velocity of the front is less than 1 and a stationary wave is not possible. For this parameter value a front of the type "fixed point–chaos" is observed (fig. 4). For ϵ close to $\frac{1}{2}$ the leading part of the front is close to the form of the stationary one. However, the difference of the velocity from 1 leads to nonuniform motion of the front and gives disturbances which grow into chaos.

For some parameter values the front has a remarkable regular structure. These are those parameter values, for which the velocity is equal to a simple ratio p/q. In this "resonant" case the front shifts p spatial positions during exactly q time steps. In the

[#2] We assume that noise does not disturb the "ground" state $u=0$, otherwise a regular front is not observed at all.

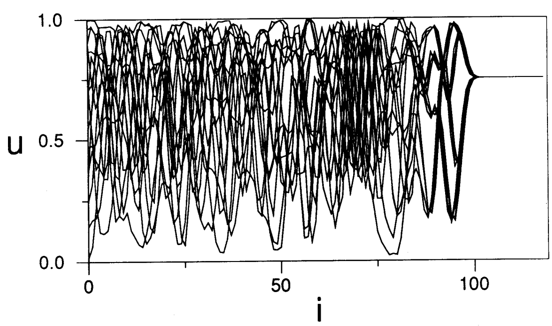

Fig. 3. Wavefront on the basis of the $u=\frac{3}{4}$ fixed point, $\epsilon=0.75$.

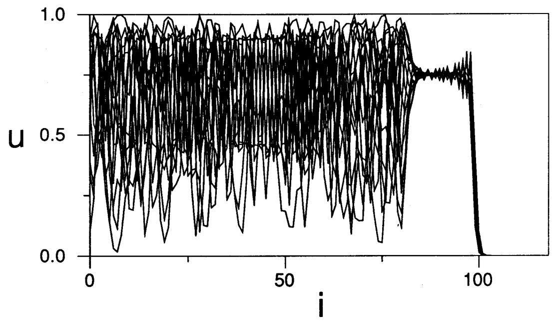

Fig. 4. Wavefront for $\epsilon=0.3$.

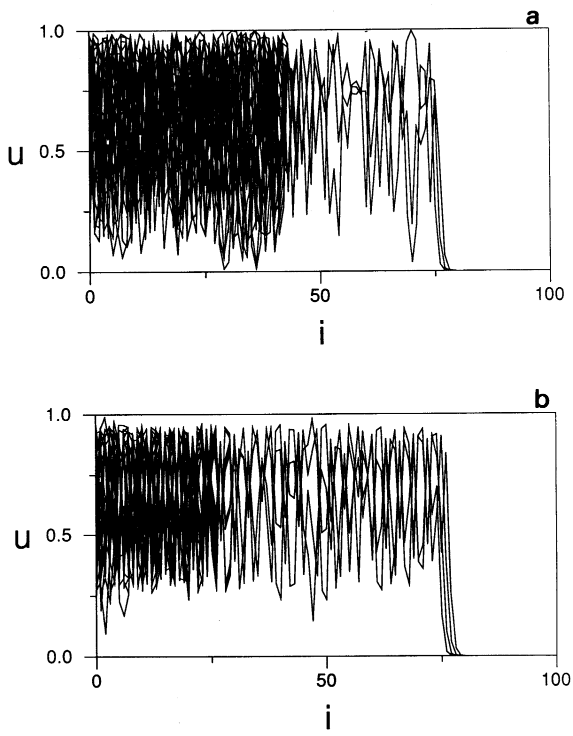

Fig. 5. Resonance wavefronts: (a) resonance $\frac{2}{3}$, $\epsilon=0.1017...$; (b) resonance $\frac{3}{4}$, $\epsilon=0.1578...$.

reference frame moving with velocity p/q the leading part of the front oscillates with period q. These oscillations, which are caused by the lattice discreteness, play the role of periodic disturbances for the back part of the front. Thus a resonance structure periodic in time appears (fig. 5). Analogous to the fixed point for $\epsilon > \frac{1}{2}$, this regular structure is convectively unstable, and small disturbances at the leading part of the front lead to chaotization of the back part. It is worth mentioning that these noisy disturbances are mainly caused not by numerical truncation errors, but by small deviations of the front velocity from a resonant value. These deviations lead to modulation of periodic resonant disturbances, and growth of this modulation evolves into chaos. It is thus clear, that "resonances" here have zero width. Another important point is that this resonant structure is regular in time, but not in space (see fig. 5). Thus, for an observer in a fixed space point chaos appears just after the leading part of the front passes. Regularity reveals itself as correlation between different points. From this point of view, at resonances the front induces equal initial perturbations at different space points, so at least temporarily the processes developing from these perturbations coincide.

In conclusion we would like to discuss whether the features of wavefront propagation in more general systems are similar to those observed in the simplest CML (1). We performed numerical calculations with different linear operators. For small diffusion constant, when the discreteness of the lattice is important, resonant regular front structures are easily observed (however, in more general models there may be no analog to the "light velocity" $S=1$). For large diffusion constants the field is smooth, and resonant front propagation gives very small periodic disturbances, so only few resonances may be observed, if any.

The author would like to thank R. Livi, L. Bunimovich, S. Ruffo and P. Grassberger for useful discussions. A part of this work was done during the author's stay at the Turin Institute for Scientific Interchanges. The author is grateful to the I.S.I. Program on "Complexity and evolution" and to the Alexander von Humboldt Stiftung for hospitality.

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