On the global scaling properties of mode-lockings in a critical circle map

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The relations between different quantitative characteristics of critical phase-lockings are studied numerically. Strong correlations are observed between the locking interval width and the sensitivity to noise and between the phase space scale and the supercriticality scale. The relation between the interval width and the phase space size displays a layered structure reproducing the Farey tree organization of rationals.

The transition to chaos through the breaking of quasiperiodic motions has been intensively studied now both theoretically and experimentally (see a recent review article [1] and references therein). Many features of this transition in a two-frequency system are adequately described by a circle mapping,

$$x_{i+1}=f(x_i)=x_i+\Omega-\frac{A}{2\pi}\sin(2\pi x_i)\ , \qquad (1)$$

which may be regarded as a transformation of the phase of one oscillator through a period of the second one. The mapping depends on two parameters: Ω describes the ratio of undisturbed frequencies while the parameter A governs the strength of the nonlinear interaction. The subcritical ($A<1$) mappings are diffeomorphisms whereas the supercritical ones ($A>1$) are non-invertible and may exhibit chaotic behavior. The borderline between these two cases consists of the critical circle mappings – homeomorphisms with one (usually cubic) inflection point; this corresponds to $A=1$ in the family (1). The properties of circle mappings of the type (1) were considered in many studies [2–5], so we shall here view only a few of them.

The dynamics of the map may be characterized by the rotation number [6]

$$\rho=\lim_{n\to\infty}[f^n(x)-x]/n\ .$$

For subcritical and critical maps this number does not depend on the initial point x. The dependence $\rho(\Omega)$ is the so-called devil's staircase, in which each rational $\rho=p/q$ is represented by an interval of Ω values (which is named the p/q-locking interval). The set of all these intervals has a full measure in the critical case [7]. The locked motion in subcritical and critical cases is represented by a stable periodic orbit of period q.

A locking region may be characterized quantitatively by its scales in the parameter space (Ω, A) and by its size in x-space. Some additional characteristics such as sensitivity to external noise [8,9] and sensitivity to coupling [10] may be also introduced. We will use the following quantities to characterize the locking region (see fig. 1). Denote by Ω_L and Ω_R the left and the right ends of the interval on the critical

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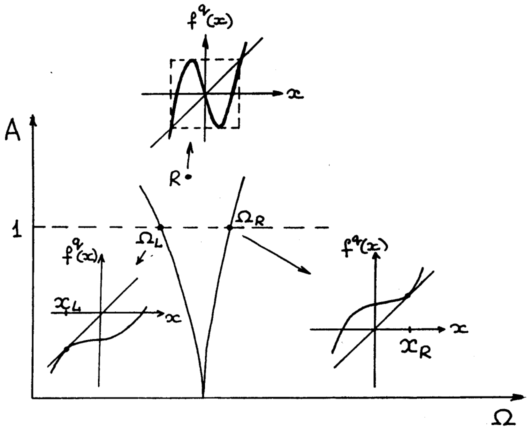

Fig. 1. Sketch of the locking region near the critical line.

line for which the p/q-locking occurs. Then $\omega = |\Omega_L - \Omega_R|$ is its natural width in the Ω-direction. For ($\Omega = \Omega_R$, $A = 1$) the mapping (1) displays a saddle-node bifurcation, at which the stable orbit collides with the unstable one. Let the point nearest to zero (which is the inflection point) of this semi-stable cycle be x_R. Similarly, for $\Omega = \Omega_L$ we determine the point x_L. The quantity $d = |x_R x_L|^{1/2}$ is a characteristic x-scale for the critical locking. In order to obtain a characteristic scale for the parameter A one has to get off the critical line $A = 1$ and to define a characteristic point above (below) it. We will use the point R (fig. 1) for which the mapping f^q has q invariant intervals. At this point the map (1) exhibits "pure chaos". The distance a from R to the critical line serves as a scale of the locking region in the variable A. We can also define, following refs. [8,9], the sensitivity Q of the critical q-periodic orbit to external random noise by

$$Q = 1 + \sum_{j=1}^{q-1} \left(\prod_{i=1+j}^{q-1} f'(x_i) \right)^2 , \quad x_1 = 0 . \tag{2}$$

Our aim in this paper is to study the relations between d, ω, a and Q for different locking regions. For some sequences of rationals these relations are known from renormalization group theory [2,3]. For example, the sequence of consecutive approximations p_m/q_m to the golden mean rotation number $(\sqrt{5}-1)/2$ displays a self-similar structure,

$$d_m \sim |\alpha|^{-m} , \quad \omega_m \sim |\delta_1|^{-m} ,$$

$$a_m \sim \delta_2^{-m} , \quad Q_m \sim \sigma^m , \tag{3}$$

where the constants $\alpha = -1.2886...$, $\delta_1 = -2.8336...$, $\delta_2 = \alpha^2$, $\sigma = 2.306...$ are obtained from the renormalization approach. The relations (3) may be written in the form $\omega \sim d^\eta$, $a \sim d^\xi Q \sim \omega^{-\nu}$ where

$$\eta = \frac{\log|\delta_1|}{\log|\alpha|} = 4.11... , \quad \xi = \frac{\log(\delta_2)}{\log|\alpha|} = 2 ,$$

$$\nu = \frac{\log|\sigma|}{\log|\delta_1|} = 0.80... .$$

The question arises whether some similar relations may be observed for the other rotation numbers and, more than that, for the whole structure of mode-lockings.

In figs. 2–4 we have plotted the calculated values of $\log(\omega)$, $\log(a)$, $\log(d)$ and $\log(Q)$ for many rational rotation numbers. One can see that the relation $a \sim d^2$ holds very well for all the rationals. This is not surprising since the relation $\delta_2 \sim \alpha^2$ is valid not only for the golden mean rotation number but for all periodic continuous fractions. In fig. 3 we observe that the response to external noise is also highly correlated to the width of the locking interval ω.

A more complicated picture is seen in fig. 4 where $\log(\omega)$ is plotted versus $\log(d)$. We see that the points do not fill the plane uniformly but display the tendency to form layers – almost parallel stretched

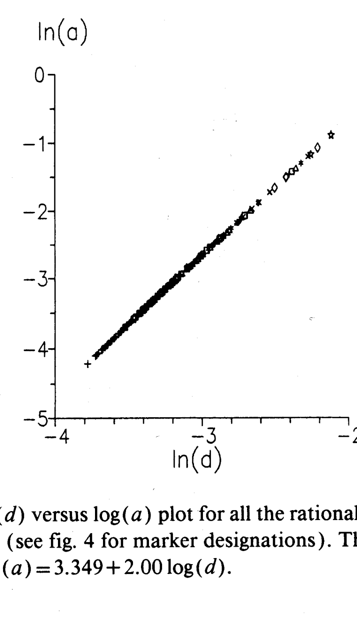

Fig. 2. Log(d) versus log(a) plot for all the rationals from Farey levels 4–10 (see fig. 4 for marker designations). The linear best fit gives $\log(a) = 3.349 + 2.00 \log(d)$.

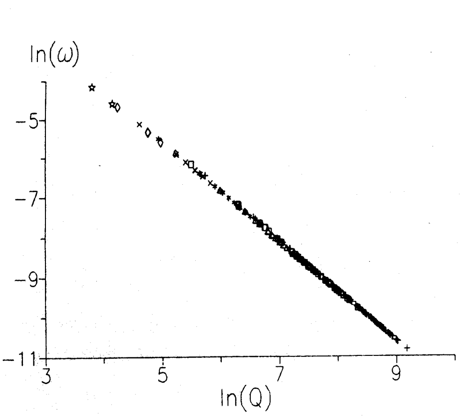

Fig. 3. Log(ω) versus log(Q) plot for the Farey levels 4–10. The linear best fit gives $\log(Q)=0.523-0.801\log(\omega)$ with the correlation coefficient 0.99993.

clouds separated by practically equal strips. At each layer the values of $\log(\omega)$ are approximately linearly related to those of $\log(d)$. The main observation of this paper is that these layers reproduce exactly the Farey-tree structure of rationals.

The well-known procedure of constructing the Farey tree is as follows [1]. Given the two first-order rationals 0/1 and 1/1 we arrive at the second-order number 1/2 as $(0+1)/(1+1)$ and then repeat the process: to obtain a new Farey level we must insert a new rational $(p_1+p_2)/(q_1+q_2)$ between each pair of already existing neighboring rationals p_1/q_1 and p_2/q_2. From the other point of view, the nth level of the Farey tree contains all the rationals ρ whose continuous fraction representation

$$\rho=1/\{m_1+1/[m_2+...+1/(m_{k-1}+1/m_k)]...\}\ ,$$

obeys the condition $\sum_1^k m_i=n$. Thus the nth level consists of 2^{n-2} rational numbers.

In fig. 4b the region containing the points in fig. 4a is redrawn in a skewed coordinate system where the last level is horizontal. Only 7 of the Farey levels are presented in the plot, however, we performed calculations for 14 levels with the same quantitative results. Thus, the values $\log(\omega)$ and $\log(d)$ for the locking from the nth Farey level are statistically connected by the relation

$$\log(\omega)\simeq C_0-bn+\kappa\log(d)\ , \tag{4}$$

where $\kappa\approx3.148$, $b\approx0.237$ and C_0 is a dimensionalization constant. For ω and d we obtain

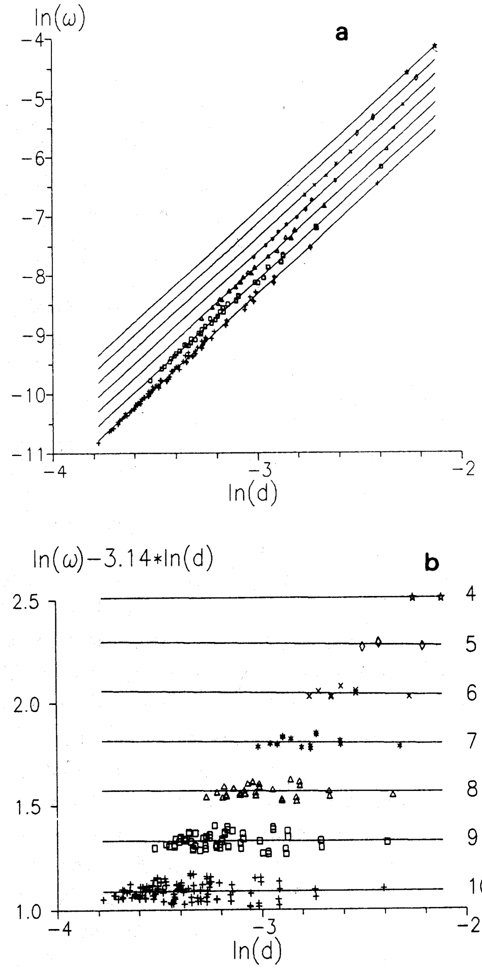

Fig. 4. Log(ω) versus log(d) plot for the Farey levels 4–10. For the linear best fit data see table 1.

$$\omega\simeq C_1B^{-n}d^{\kappa}\ , \tag{5}$$

where $C_1=\exp(C_0)$, $B=\exp(b)=1.267$. The question remains whether the relations above are universal. We calculated the parameters of lockings for the family of critical mappings

$$x_i=f(x_i)$$
$$=x_i+\Omega-\frac{3U+1}{2\pi}\sin(2\pi x_i)+\frac{U}{2\pi}\sin(6\pi x_i)$$
$$(-\tfrac{1}{3}<U<\tfrac{1}{24})\ . \tag{6}$$

The layered structure in the plane $\log(\omega)$–$\log(d)$

Table 1
The coefficients of the linear best fit $\log(\omega_n)=C_n+\kappa_n\log(d_n)$ for the different Farey levels; $b_n=C_n-C_{n-1}$.

Farey level n	Linear best fit C_n	κ_n	b_n	Correlation coefficient
4	2.499	3.137		1.0
5	2.242	3.126	0.256	0.9996
6	2.005	3.126	0.237	0.9993
7	1.776	3.130	0.229	0.9991
8	1.549	3.134	0.227	0.9989
9	1.321	3.137	0.228	0.9988
10	1.091	3.140	0.230	0.9987
11	0.860	3.142	0.231	0.9986
12	0.627	3.144	0.233	0.9986
13	0.393	3.145	0.234	0.9986
14	0.157	3.146	0.235	0.9985

persists but with slightly varying constants κ and b: $3.07<\kappa(U)<3.26$, $0.22<b(U)<0.24$. (For $U=\frac{1}{24}$ when the inflection of $f(x)$ is of fifth order, $\kappa=4.628$, $b=0.289$.) At the same time it should be noted that the increments from layer to layer of the values $\log(\omega)$ and $\log(d)$ averaged over each level seem to be independent of U. (It is noteworthy that the difference between the average values $\langle\log(\omega)\rangle^{(j-1)}$ and $\langle\log(\omega)\rangle^{(j)}$, computed for the $(j-1)$st and jth levels, tends with growth of j to the universal limit 0.871... which is practically indistinguishable from numerical estimates of the fractal dimension of the devil's staircase [5].)

There seems to be some arbitrariness in our selection of the characteristic scales of the mode-lockings. We have tried other choices as well, taking for d the largest distance on the circle between two neighboring points of the neutrally stable periodic orbit and for the transcritical parameter scale a either the shortest distance between the critical line and the line of period-doubling bifurcation, or the depth into the subcritical region, at which the width of the locked region is half of the critical width ω. In all cases the results were qualitatively the same; the asymptotical values of the scaling constants within numerical accuracy coincided with the above data.

One may also suggest that the above results should be ascribed to the convenient parametrization of the family (1), in which the parameters responsible for rotation and nonlinear interaction are decoupled. Our calculations for reparametrizations of this family as well as for some other families (in which the analog of Ω is included not additively), provide evidence that only the first levels of the Farey tree are influenced, whereas at lower levels (where the characteristic quantities become smaller) the above scaling phenomena persist.

In conclusion, we have obtained relations between the different characteristics of mode-lockings of the critical circle map. The results allow one to predict with rather high accuracy the parameters of rational locking intervals from the high Farey levels, provided the properties of the first few lockings are known. The connection of the picture above with the Farey-tree renormalization scheme [11–14] remains an open question.

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