On the global scaling properties of mode-lockings in a critical circle map

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The relations between different quantitative characteristics of critical phase-lockings are studied numerically. Strong correlations are observed between the locking interval width and the sensitivity to noise and between the phase space scale and the supercriticality scale. The relation between the interval width and the phase space size displays a layered structure reproducing the Farey tree organization of rationals.

The transition to chaos through the breaking of quasiperiodic motions has been intensively studied now both theoretically and experimentally (see a recent review article [1] and references therein). Many features of this transition in a two-frequency system are adequately described by a circle mapping,

\[ x_{i+1} = f(x_i) = x_i + \Omega - \frac{A}{2\pi} \sin(2\pi x_i), \]

which may be regarded as a transformation of the phase of one oscillator through a period of the second one. The mapping depends on two parameters: \( \Omega \) describes the ratio of undisturbed frequencies while the parameter \( A \) governs the strength of the nonlinear interaction. The subcritical (\( A < 1 \)) mappings are diffeomorphisms whereas the supercritical ones (\( A > 1 \)) are non-invertible and may exhibit chaotic behavior. The borderline between these two cases consists of the critical circle mappings - homeomorphisms with one (usually cubic) inflection point; this corresponds to \( A = 1 \) in the family (1). The properties of circle mappings of the type (1) were considered in many studies [2–5], so we shall here view only a few of them.

The dynamics of the map may be characterized by the rotation number \( \rho \)

\[ \rho = \lim_{n \to \infty} \frac{f^n(x) - x}{n}. \]

For subcritical and critical maps this number does not depend on the initial point \( x \). The dependence \( \rho(\Omega) \) is the so-called devil's staircase, in which each rational \( \rho = p/q \) is represented by an interval of \( \Omega \) values (which is named the \( p/q \)-locking interval). The set of all these intervals has a full measure in the critical case [7]. The locked motion in subcritical and critical cases is represented by a stable periodic orbit of period \( q \).

A locking region may be characterized quantitatively by its scales in the parameter space (\( \Omega, A \)) and by its size in \( x \)-space. Some additional characteristics such as sensitivity to external noise [8, 9] and sensitivity to coupling [10] may be also introduced. We will use the following quantities to characterize the locking region (see fig. 1). Denote by \( \Omega_l \) and \( \Omega_r \) the left and the right ends of the interval on the critical
line for which the \( p/q \)-locking occurs. Then \( \omega = |\Omega_L - \Omega_R| \) is its natural width in the \( \Omega \)-direction. For \( (\Omega = \Omega_R, \ A=1) \) the mapping (1) displays a saddle-node bifurcation, at which the stable orbit collides with the unstable one. Let the point nearest to zero (which is the inflection point) of this semi-stable cycle be \( x_R \). Similarly, for \( \Omega = \Omega_L \) we determine the point \( x_L \). The quantity \( d = |x_R - x_L|^{1/2} \) is a characteristic \( x \)-scale for the critical locking. In order to obtain a characteristic scale for the parameter \( A \) one has to get off the critical line \( A = 1 \) and to define a characteristic point above (below) it. We will use the point \( R \) (fig. 1) for which the mapping \( f^q \) has \( q \) invariant intervals. At this point the map (1) exhibits "pure chaos". The distance \( a \) from \( R \) to the critical line serves as a scale of the locking region in the variable \( A \). We can also define, following refs. [8,9], the sensitivity \( Q \) of the critical \( q \)-periodic orbit to external random noise by

\[
Q = 1 + \sum_{j=1}^{q-1} \left( \prod_{i=1}^{q-1} f'(x_i) \right)^2, \quad x_1 = 0. \tag{2}
\]

Our aim in this paper is to study the relations between \( d, \omega, a \) and \( Q \) for different locking regions. For some sequences of rationals these relations are known from renormalization group theory [2,3]. For example, the sequence of consecutive approximations \( p_m/q_m \) to the golden mean rotation number \((\sqrt{5}-1)/2\) displays a self-similar structure,

\[
d_m \sim |\alpha|^{-m}, \quad \omega_m \sim |\delta_1|^{-m}, \tag{3}
\]

where the constants \( \alpha = -1.2886..., \delta_1 = -2.8336..., \delta_2 = \alpha^2, \sigma = 2.306... \) are obtained from the renormalization approach. The relations (3) may be written in the form \( \omega \sim d^n, a \sim d^c \) where

\[
\eta = \frac{\log |\delta_1|}{\log |\alpha|} = 4.11..., \quad \xi = \frac{\log (\delta_1)}{\log |\alpha|} = 2, \\
\nu = \frac{\log |\sigma|}{\log |\delta_1|} = 0.80... .
\]

The question arises whether some similar relations may be observed for the other rotation numbers and, more than that, for the whole structure of mode-lockings.

In figs. 2–4 we have plotted the calculated values of \( \log(\omega), \log(a), \log(d) \) and \( \log(Q) \) for many rational rotation numbers. One can see that the relation \( a \sim d^2 \) holds very well for all the rationals. This is not surprising since the relation \( \delta_1 \sim \alpha^2 \) is valid not only for the golden mean rotation number but for all periodic continuous fractions. In fig. 3 we observe that the response to external noise is also highly correlated to the width of the locking interval \( \omega \).

A more complicated picture is seen in fig. 4 where \( \log(\omega) \) is plotted versus \( \log(d) \). We see that the points do not fill the plane uniformly but display the tendency to form layers – almost parallel stretched
clouds separated by practically equal strips. At each layer the values of \( \log(\omega) \) are approximately linearly related to those of \( \log(d) \). The main observation of this paper is that these layers reproduce exactly the Farey-tree structure of rationals.

The well-known procedure of constructing the Farey tree is as follows [1]. Given the two first-order rationals 0/1 and 1/1 we arrive at the second-order number 1/2 as \((0+1)/(1+1)\) and then repeat the process: to obtain a new Farey level we must insert a new rational \((p_1 + p_2)/(q_1 + q_2)\) between each pair of already existing neighboring rationals \(p_1/q_1\) and \(p_2/q_2\). From the other point of view, the \(n\)th level of the Farey tree contains all the rationals \(\rho\) whose continuous fraction representation

\[
\rho = 1/[m_1 + 1/[m_2 + ... + 1/(m_{k-1} + 1/m_k)]...],
\]

obeys the condition \(\sum_k m_k = n\). Thus the \(n\)th level consists of \(2^{n-2}\) rational numbers.

In fig. 4b the region containing the points in fig. 4a is redrawn in a skewed coordinate system where the last level is horizontal. Only 7 of the Farey levels are presented in the plot, however, we performed calculations for 14 levels with the same quantitative results. Thus, the values \(\log(\omega)\) and \(\log(d)\) for the locking from the \(n\)th Farey level are statistically connected by the relation

\[
\log(\omega) = C_0 - b n + \kappa \log(d),
\]

where \(\kappa \approx 3.148\), \(b \approx 0.237\) and \(C_0\) is a dimensionalization constant. For \(\omega\) and \(d\) we obtain

\[
\omega \approx C_1 B^{-n} d^\kappa,
\]

where \(C_1 = \exp(C_0), B = \exp(b) = 1.267\). The question remains whether the relations above are universal. We calculated the parameters of lockings for the family of critical mappings

\[
x_i = f(x_i)
\]

\[
= x_i + \Omega - \frac{3U+1}{2\pi} \sin(2\pi x_i) + \frac{U}{2\pi} \sin(6\pi x_i)
\]

\((-\frac{1}{2} < U < \frac{1}{2})\).
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References