J. Phys. A: Math. Gen. 24 (1991) 4587-4597. Printed in the UK

Symmetry breaking bifurcation for coupled chaotic attractors

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Received 20 May 1991

Abstract. We consider transitions from synchronous to asynchronous chaotic motion in two identical dissipatively coupled one-dimensional mappings. We show that the probability density of the asymmetric component satisfies a scaling law. The exponent in this scaling law varies continuously with the distance from the bifurcation point, and is determined by the spectrum of local Lyapunov exponents of the uncoupled map. Finally we show that the topology of the invariant set is rather unusual: though the attractor for supercritical coupling is a line, it is surrounded by a strange invariant set which is dense in a two-dimensional neighbourhood of the attractor.

1. Introduction

A natural path leading from simple dynamical systems to more complicated ones consists in coupling several identical copies of the simple system. This idea has been exploited in [1-5], where only small numbers of coupled systems were considered. Systems of very many coupled maps arranged in regular lattices are known as coupled map lattices [7], but will not be considered here.

The papers [1, 2, 6] focus on dissipative couplings which tend to synchronize the subsystems. In [1, 2] it was shown that for strong dissipative coupling a synchronous regime is reached where the subsystems move completely in phase and with the same amplitude. With decreasing coupling strength, this regime looses stability and the instantaneous states of the subsystems begin to differ. The instability threshold was shown in [1, 2] to depend on the Lyapunov number of the chaotic motion. In [8] it was proposed to use this for experimental estimates of the Lyapunov number, by actually coupling two identical systems. The loss of stablity of the synchronous regime can be considered as a spontaneous symmetry breakdown, at least as far as the symmetry of the instantaneous state is concerned. In a statistical sense, the symmetry will in general be preserved even in the broken phase. This is seen in figure 1, where the distribution is symmetric in the asynchronous state. For experimental observation of this bifurcation see [8].

Detailed theories of the symmetry breaking bifurcation were developed in [2, 9, 10]. In [2], a mean-field approximation was used to show that both hard and soft transitions may occur. A hard transition was observed, for example, in [1] for interacting Lorenz attractors. For a special case of interacting logistic maps a nonlinear statistical theory was developed in [9]. This theory describes well the intermittent properties

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0305-4470/91/194587+11$03.50

observed in the numerical experiments of [11] and gives a Gaussian distribution for the asymmetrical component. However, this theory cannot be directly applied to more general chaotic attractors, where power-law distributions of the asymmetrical component are usually observed [11].

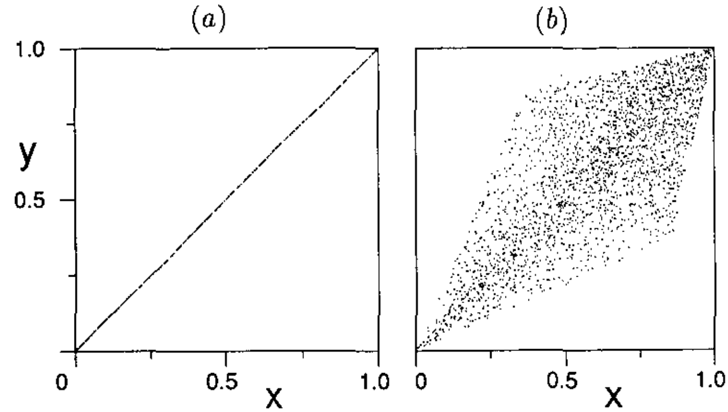

Figure 1. Attractor in the system (1), (7) with $a=\frac{2}{3}$. For this system, $\sigma_c=0.2354\ldots$. (*a*): $\sigma=\sigma_c+0.05$; panel (*b*): $\sigma=\sigma_c-0.05$.

In this paper we present a theory which indeed gives such a power-law distribution for the asymmetrical component (section 2), and test it for coupled 1D maps. It turns out that the exponent depends on the spectrum of local Lyapunov exponents [12, 13] in a non-trivial way. While this power law is obtained already in linear stability analysis, we also give a full nonlinear analysis in section 4. Before that, we point out that in section 3 a number of very unusual topological features of the attractor.

2. Linear stability analysis

The systems we consider throughout this paper can be written as

$$\begin{aligned} x_{n+1} &= f(x_n)+\sigma(f(y_n)-f(x_n)) \\ y_{n+1} &= f(y_n)+\sigma(f(x_n)-f(y_n)) \end{aligned} \tag{1}$$

where f is some nonlinear function which governs the dynamics of the single system, and σ is an interaction constant. It is convenient to introduce the variables

$$u_n=\frac{x_n+y_n}{2} \qquad v_n=\frac{x_n-y_n}{2}. \tag{2}$$

Then (1) reduces to

$$\begin{aligned} u_{n+1} &= \tfrac{1}{2}(f(u_n+v_n)+f(u_n-v_n)) \\ v_{n+1} &= (\tfrac{1}{2}-\sigma)(f(u_n+v_n)-f(u_n-v_n)). \end{aligned} \tag{3}$$

One can easily see from (3) that for this type of coupling a synchronous state $v_n\equiv 0$, $u_{n+1}=f(u_n)$ exists for all σ. Linearizing near this state we obtain

$$u_{n+1}=f(u_n) \qquad v_{n+1}=(1-2\sigma)f'(u_n)v_n \tag{4}$$

or, for the variable $z_n = \ln|v_n|$,

$$z_{n+1} = z_n + \ln(1-2\sigma) + \ln|f'(u_n)|. \tag{5}$$

From (5) one immediately obtains the stability threshold σ_c as [1, 2]

$$\ln(1-2\sigma_c) + \lambda = 0 \tag{6}$$

where

$$\lambda = \langle \ln|f'(u)| \rangle = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(u_i)|$$

is the Lyapunov exponent for the uncoupled system. For $\sigma < \sigma_c$ an asynchronous state arises, as is illustrated in figure 1 with the skewed tent map

$$f(x) = \begin{cases} x/a & \text{if } 0 \leqslant a \\ (1-x)/(1-a) & \text{if } a < x \leqslant 1. \end{cases} \tag{7}$$

We will consider (3) near this threshold, so it is convenient to use the bifurcation parameter $\varepsilon = \ln|1-2\sigma| + \lambda$ which vanishes at the threshold (6), and to rewrite (5) in the form

$$z_{n+1} = z_n + \varepsilon + \ln|f'(u_n)| - \lambda. \tag{8}$$

If the trajectory u_n is chaotic, the increments $z_{n+1} - z_n$ are essentially random variables. They are of course not independent, but the correlations between them will in general decay exponentially. Thus we can expect that the averaged quantities

$$\Lambda_k = \frac{1}{N} \sum_{n=kN}^{(k+1)N-1} \ln|f'(u_n)| \tag{9}$$

become independent for large N. Also, these quantities (which can be considered as 'local' or 'effective' Lyapunov exponents [14, 15]) should satisfy a central limit theorem for $N \to \infty$. This implies for their probability distribution $p(\Lambda; N)$ an ansatz [13]

$$p(\Lambda; N) \sim \mathrm{e}^{N\phi(\Lambda)} \tag{10}$$

and for its generating function [12]

$$\int \mathrm{d}\Lambda\, p(\Lambda; N)\, \mathrm{e}^{sN\Lambda} \approx \mathrm{e}^{Ng(s)} \tag{11}$$

with $g(s)$ and $\phi(\Lambda)$ being mutual Legendre transforms,

$$\phi(\Lambda) + g(s) = s\Lambda \qquad \Lambda = \frac{\mathrm{d}g}{\mathrm{d}s} \qquad s = \frac{\mathrm{d}\phi}{\mathrm{d}\Lambda}. \tag{12}$$

We will now show that (8)–(12) imply a power law for the distribution of $|v|$. From (8) we obtain

$$z_{(k+1)N} = z_{kN} + N(\varepsilon - \lambda) + N\Lambda_k. \tag{13}$$

According to our assumptions, the correlations between z_{kN} and Λ_k can be neglected. Thus the probability distributions $W_k(z)$ and $W_{k+1}(z)$ for z_{kN}, respectively $z_{(k+1)N}$, satisfy [16]

$$W_{k+1}(z) = \int \mathrm{d}\Lambda\, p(\Lambda; N)\, W_k(z - N(\varepsilon - \lambda) - N\Lambda). \tag{14}$$

We look for a stationary distribution. If we try an ansatz

$$W_k(z) = W_{k+1}(z) = \mathrm{e}^{\kappa z} \tag{15}$$

then we get

$$\mathrm{e}^{(\varepsilon-\lambda)\kappa N} = \int \mathrm{d}\Lambda\, p(\Lambda; N)\, \mathrm{e}^{-\kappa N \Lambda} \approx \mathrm{e}^{Ng(-\kappa)}. \tag{16}$$

In terms of the variable $|v| = \mathrm{e}^z$, this gives a power law distribution

$$P_v(|v|) \propto |v|^{\kappa-1} \tag{17}$$

with κ obtained implicitly from

$$g(-\kappa) + (\lambda - \varepsilon)\kappa = 0. \tag{18}$$

As for any perfect power law, this distribution is not normalizable. Indeed, since $g(0) = 0$ and $g'(s)$ is monotonic with $g'(0) = \lambda$, we see that κ changes sign exactly at the bifurcation point $\varepsilon = 0$, so that the distribution diverges for $|v| \to \infty$ if $\sigma < \sigma_c$, and for $v \to 0$ if $\sigma > \sigma_c$. To obtain a normalizable distribution, we have to take into account that (7) has to break down at large values of z (i.e. large values of v). It will also break down at large negative values of z (i.e. for $v \to 0$) if the symmetry between the two systems is slightly broken.

The symmetry breaking can be done explicitly, by considering slightly non-identical systems [2, 17], or implicitly by adding noise to (1).

Let us assume that the uncoupled systems depend on a parameter b, and let us consider instead of (1) two coupled systems with slightly different parameters $b \pm \gamma$:

$$\begin{aligned} x_{n+1} &= f(x_n, b+\gamma) + \sigma(f(y_n, b-\gamma) - f(x_n, b+\gamma)) \\ y_{n+1} &= f(y_n, b-\gamma) + \sigma(f(x_n, b+\gamma) - f(y_n, b-\gamma)). \end{aligned} \tag{19}$$

Then instead of (4) we obtain

$$v_{n+1} = (1-2\sigma)\left[\frac{\partial f(u_n, b)}{\partial u} v_n + \frac{\partial f(u_n, b)}{\partial b} \gamma\right]. \tag{20}$$

For $|v_n| \leqslant \gamma$ the second term in (20) dominates, thus giving a lower cutoff at $|v| \approx \gamma$. A similar cutoff is provided if we replace (1) by

$$\begin{aligned} x_{n+1} &= f(x_n) + \sigma(f(y_n) - f(x_n)) + r_n \\ y_{n+1} &= f(y_n) + \sigma(f(x_n) - f(y_n)) + s_n \end{aligned} \tag{21}$$

where r_n and s_n are independent Gaussian random noises with $\langle r_n r_m \rangle = \langle s_n s_m \rangle = \delta_{nm}\gamma$.

While the cutoff at small $|v|$ is not intrinsic to the system, the cutoff at large $|v|$ is unavoidable because of terms in (3) which are nonlinear in v_n. It depends on the precise form of the mapping f. For the logistic map at fully developed chaos ($f(x) = 4x(1-x)$), a nonlinear stability analysis was developed in [9]. In the following we shall present a similar theory for the piecewise-linear map given in (7), and compare it with direct simulations. But before doing this, we have to discuss some geometric and topological aspects related to this problem.

3. Geometrical and topological aspects

In this section, we shall concentrate on the subcritical case depicted in figure 1(a), with $\lambda > -\ln(1-2\sigma) < \Lambda_{max}$. Here Λ_{max} is the supremum of the support of $\phi(\Lambda)$, which coincides in general with the Lyapunov exponent of the most unstable periodic cycle of the map $f(x)$. We have argued in section 2 that for such couplings there exists an (albeit non-normalizable) smooth invariant measure outside the diagonal $x=y$. This would indicate that there exists a strange invariant set which is dense in some two-dimensional neighbourhood of the diagonal.

Indeed, we shall see that the situation is not at all as trivial as might be suggested by figure 1(a). In particular, as we will see below, definitions of attractors based on topological concepts [18-21] would not give the seemingly obvious result that the attractor in this case is the diagonal $x=y$. This is only obtained if one adopts the definition of [22] based on measure-theoretic concepts. Even then, the attractor has very strange properties. We will argue that:

(i) points on the attractor for which the unstable manifold is two-dimensional are dense on it;

(ii) for all points on the attractor (except $(x,y)=(0,0)$ and $(1,1)$) there exist neighborhoods in which periodic points and points diverging exponentially from it are dense.

Though these results should hold more generally, we shall discuss here only the skew tent map of (7) with $a>\frac{1}{2}$. In this case, the average Lyapunov exponent is

$$\lambda = -a \ln a - (1-a)\ln(1-a) \tag{22}$$

while the fixed point $x_{fp} = 1/(2-a)$ has Lyapunov exponent

$$\lambda_{fp} = -\ln(1-a) > \lambda. \tag{23}$$

Correspondingly, global stability is lost at

$$\sigma_c = \frac{1-e^{-\lambda}}{2} \tag{24}$$

while the fixed point loses stability only at

$$\sigma_{c,fp} = \frac{1-e^{-\lambda_{fp}}}{2} = \frac{a}{2} > \sigma_c. \tag{25}$$

Let us now chose σ such that $\sigma_c < \sigma < \sigma_{c,fp}$. This is the case shown in figure 1(a), with a seemingly one-dimensional attractor. But in its fixed point $(u, v) = (x_{fp}, 0)$, the coupled system has two positive Lyapunov exponents and thus a two-dimensional unstable manifold. The same will hold for all preimages of this point, which fill the diagonal in figure 1(a) densely. This proves our first point.

Moreover, one easily sees that a line $\{u, v \mid u = x_{fp}, -v_1 < v < v_1\}$ with sufficiently small v_1 will be mapped onto a similar line $\{u, v \mid u = x_{fp}, -v_2 < v < v_2\}$ with $v_2 > v_1$. Thus the set of all its forward iterates is an invariant C^0-manifold of the fixed point. In figure 2 we show the first few iterates of such a line for $a=\frac{2}{3}$ and $\sigma = (\sigma_c + \sigma_{c,fp})/2$. They seem to start filling up the rhombus $ABCD$. This is confirmed by much longer iterates which fill the rhombus densely within the accuracy of the figure. Indeed this rhombus, with a point A defined by $(x_A, y_A) = (2\sigma, (1-2\sigma+2\sigma^2)/(1-\sigma))$ and C by $(x_C, y_C) = (y_A, x_A)$, is easily checked to be invariant. The invariant polygon shown in

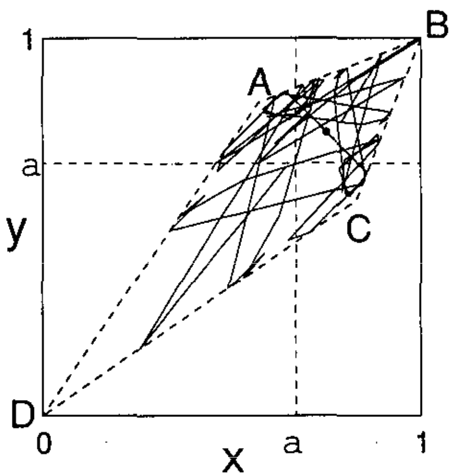

Figure 2. First 17 images of the interval $\{u = x_{fp},\ 0 \leq v \leq 0.01\}$ in the system defined by (1), (7) with $a = \frac{2}{3}$, $\sigma = (\sigma_c + \sigma_{c,fp})/2$. The rhombus *ABCD* with $x_A = 2\sigma$, $y_A = (1 - 2\sigma + 2\sigma^2)(1-\sigma)^{-1}$, $x_C = y_A$, $y_C = x_A$ is mapped onto itself.

figure 2 and its preimages contain the dense set of points diverging from the attractor mentioned in the second item above. We might add that similar sets arise from all other unstable periodic orbits of $f(x)$ whose Lyapunov exponent is such that their σ_c is larger than σ. These points are again dense.

Finally, we computed all periodic orbits of the coupled system with period up to $p = 16$ (see figure 3). We found that *all* asymmetric cycles (i.e. cycles with $x_n \neq y_n$) are of node or of focus type, i.e. they have two-dimensional unstable manifolds. As seen in figure 3, the distribution of cycles is very non-uniform. Nevertheless, by comparing orbit sets with different periods we conjecture that they become dense in the rhombus *ABCD* for $p \to \infty$. This would indeed be expected if the omega limit set is dense in

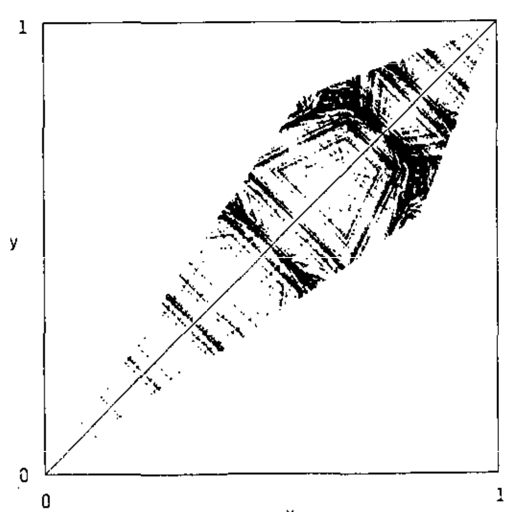

Figure 3. All periodic orbits of (1), (7) with period $p \leq 16$. For increasing p, these points seem to fill the rhombus *ABCD* of figure 2 densely.

this rhombus. For large p, the density of periodic points should become proportional to the density of the maximum entropy measure of the coupled system. The non-uniformity seen in figure 3 corresponds then to strong multifractality of this measure. We might add that the numbers N_p of points on p-periodic asymmetric orbits for $p = 2, 4, 6, \ldots, 16$ are 6, 30, 130, 454, 1676, 7236, 28 454 and 110 678. This suggests that the topological entropy of asymmetric orbits is $\approx \ln 2$, close to the entropy of symmetric orbits.

Let us now compare this with attractor definitions found in the literature. For instance, in [19] an attractor was defined as a closed subset $\mathscr{A}$ of the non-wandering set Ω which has a neighbourhood U such that $\bigcap_{n>0} f^n U = \mathscr{A}$. With this definition obviously the rhombus $ABCD$ would be the attrctor in the present case. The fact that this definition is too strong was already observed earlier, e.g. since it would not apply to the Feigenbaum attractor which has unstable periodic orbits arbitrarily close to it, so that no open neighbourhood could be completely attracted.

The definition which indeed gives the diagonal $x = y$ as attractor is the one by Milnor [22], who essentially demands that $\mathscr{A}$ is a closed set for which:

(i) the *realm of attraction* $\rho(\mathscr{A})$ has strictly positive measure, and

(ii) there is no strictly smaller closed set $\mathscr{A}' \subset \mathscr{A}$ whose realm of attraction coincides with $\rho(\mathscr{A})$ up to Lebesgue measure zero.

The realm of attraction is essentially what is usually called the basin of attraction, namely the set of all points whose omega limit set is in $\mathscr{A}$.

The essential feature here is that it is not demanded that the realm of attraction is an open set or contains any open neighbourhoods. The only thing we still have to prove is that the omega limit sets of most points with $x \approx y$ are on the diagonal. *A priori*, it could be that most trajectories make rare and short excursions away from it which would not be seen in a picture like figure 1(a). We can exclude this if we accept the linear stability analysis of the last section. From (8) and (10) one can easily deduce that the probability $p(v_n > v^*)$, $v^* > 0$ for a randomly chosen initial point decreases exponentially with n. The probability of a later excursion satisfies $P(v_n > v^*) < \Sigma_{m>n}\, p(v_m > v^*)$ and tends thus also to zero for $n \to \infty$.

4. Nonlinear stability analysis

Let us now come back to describing the invariant probability distribution of v in the subcritical case where symmetry is broken (figure 1(b)). We again discuss only the tent map (7) with $a > \frac{1}{2}$. Inserting it into (3), we find the following. If u_n lies outside the interval $I_n = (a - |v_n|, a + |v_n|)$, the linearized equation (4) remains valid. But if u_n falls into the interval I_n (this occurs approxiately with probability $|2v_n|$), equation (4) is modified. The ratio $|v_{n+1}/v_n|$ is a piecewise linear function of u_n, shown in figure 4. Let us neglect the small influence of v on the dynamics of u and assume that the density of u_n is uniform as it would be for the uncoupled map. Then, with probability $1 - 2|v_n|$ the linearized equation (4) is valid (this corresponds to the part of figure 4 outside I_n), and with probability $2|v_n|$ the right-hand side of (4) is additionally multiplied by a random quantity ρ, uniformly distributed between 0 and 1 (this corresponds to the part of figure 4 inside I_n). Thus, within these assumptions we obtain for $z_n = \ln|v_n|$ the equation

$$z_{n+1} = z_n + \eta_n + \xi_n q(z_n) \qquad (26)$$

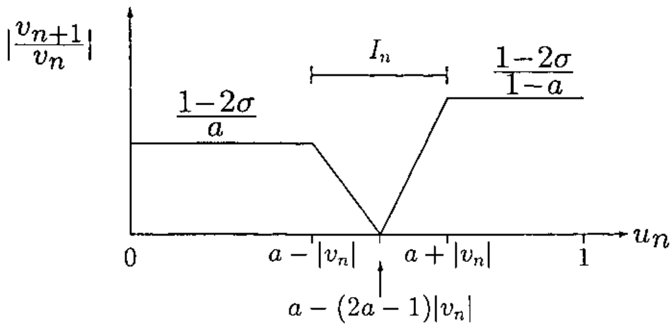

Figure 4. Sketch of v_{n+1}/v_n against u_n for fixed v_n. Only outside the interval I_n, v_{n+1} depends linearly on v_n.

where η_n and ξ_n are random variables with densities $Q(\eta)$ (respectively $P(\xi)$) given by

$$\begin{aligned} &Q(\eta)=a\delta(\eta-\ln|(1-2\sigma)/a|)+(1-a)\delta(\eta-\ln|(1-2\sigma)/(1-a)|) \\ &P(\xi)=\mathrm{e}^{\xi} \qquad -\infty<\xi<0 \end{aligned} \tag{27}$$

and $q(z_n)$ is a random variable which assumes the values $q(z_n)=0$ with probability $1-2|v_n|=1-2\mathrm{e}^{z_n}$, and $q(z_n)=1$ with probability $2|v_n|=2\mathrm{e}^{z_n}$. The probability distribution $W_n(z)$ obeys thus an equation

$$\begin{aligned} W_{n+1}(z)=&\int W_n(z-\eta)Q(\eta)(1-2\mathrm{e}^{z-\eta})\,\mathrm{d}\eta \\ &+\iint W_n(z-\eta-\xi)Q(\eta)P(\xi)\,2\mathrm{e}^{z-\eta-\xi}\,\mathrm{d}\eta\,\mathrm{d}\xi. \end{aligned} \tag{28}$$

This can be simplified to

$$W_{n+1}(z)=\int\left\{W_n(z-\eta)+2\frac{d}{dz}\left[\mathrm{e}^{z-\eta}\int_{z-\eta}^{\infty}W_n(x)\,\mathrm{d}x\right]\right\}Q(\eta)\,\mathrm{d}\eta. \tag{29}$$

Let us look for the asymptotic behaviour of $W(z)$ for large negative z. Assuming that in this domain $W(z)\sim\exp(\kappa z)$, we obtain for κ

$$\int\left[\exp(-\kappa\eta)+2\exp(z(1-\kappa)-\eta)-\frac{2}{\kappa}\exp(z(2-\kappa)-2\eta)\right]Q(\eta)\,\mathrm{d}\eta=1. \tag{30}$$

For $\kappa<1$ the second and the third terms in eq. (30) may be neglected and from eqs. (30), (27) follows

$$a\exp(-\kappa\ln|(1-2\sigma)/a|)+(1-a)\exp(-\kappa\ln|(1-2\sigma)/(1-a)|)=1. \tag{31}$$

This is just the equation that follows from (18) if one uses for the tent map the scaling function $\phi(\Lambda)$ derived in [13]. Equation (31) is valid, however, only for $\kappa<1$. Otherwise the second term in (30) cannot be neglected. This happens for sufficiently small σ where one finds $\kappa=1$, i.e. $W(z)\sim\mathrm{e}^z$. This corresponds to a non-singular and smooth probability distribution $P_v(|v|)\approx$ constant at $|v|\approx 0$. Equation (31) was checked by direct simulations. As seen in figure 5, agreement is very good.

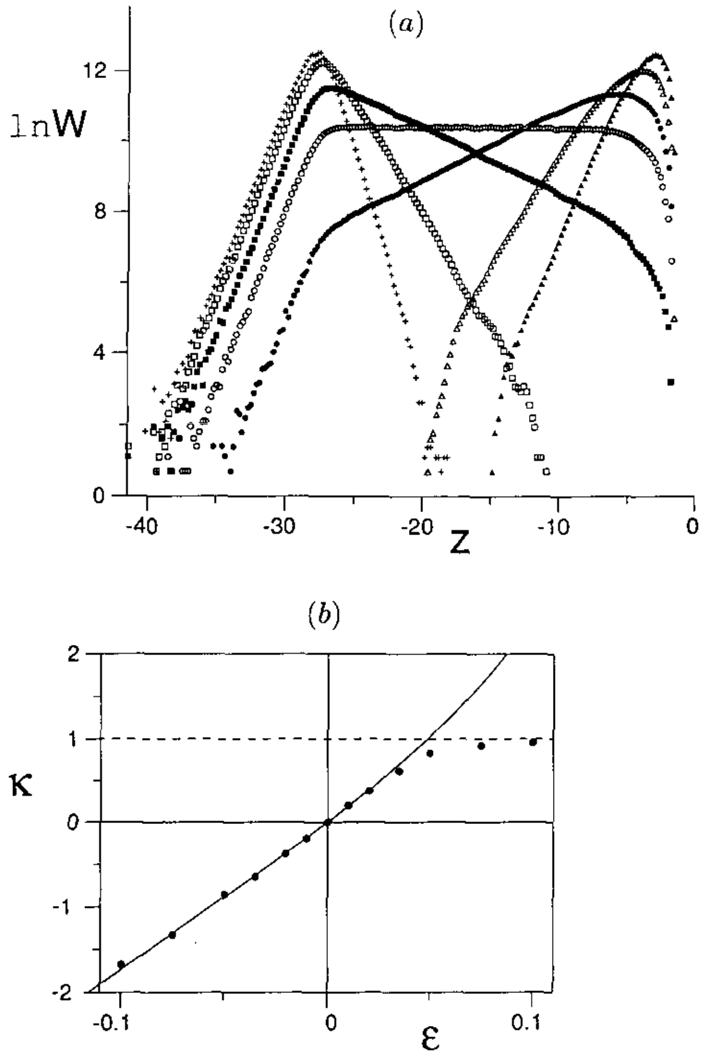

Figure 5. (a) Probability density (in arbitrary units) of the variable z in the system given by eqs. (19), (7) for $a=\frac{2}{3}$, $\gamma=10^{-12}$ and for different values of coupling constant: crosses: $\varepsilon=-0.075$, open squares: $\varepsilon=-0.035$, filled squares: $\varepsilon=-0.01$, open circles: $\varepsilon=0$, filled circles: $\varepsilon=0.01$, open triangles: $\varepsilon=0.035$, filled triangles; $\varepsilon=0.1$. (b) Slopes of the curves in panel (a) in the central region where these curves are linear (circles), compared with theoretical prediction (31) (solid curve).

Unfortunately, we were not able to get a closed solution of (29) in the general case. We can solve (29) only in the symmetrical case $a=\frac{1}{2}$, where the probability density shows no power-law asymptotics since $\phi(\Lambda)=\delta(\Lambda)$. For $a=\frac{1}{2}$

$$Q(\eta)=\delta(\eta-\ln|2(1-2\sigma)|)=\delta(\eta-\varepsilon).$$

Substituting into (29) we get

$$W_{n+1}(z+\varepsilon)=W_n(z)(1-2e^z)+2e^z\int_z^{\infty}W_n(x)\,dx$$

For $\varepsilon \ll 1$ we may write $W(z+\varepsilon) \approx W(z) + \varepsilon\, \mathrm{d}W/\mathrm{d}z$ and obtain the following invariant distribution

$$W(z) = \frac{2}{\varepsilon} \exp\left(z - \frac{2}{\varepsilon} e^{z}\right) \qquad \text{or} \qquad P_v(|v|) = \frac{2}{\varepsilon} e^{-(2/\varepsilon)|v|}. \tag{32}$$

We compare solution (32) with numerical results in figure 6.

5. Conclusions

We have shown that at the symmetry breaking bifurcation in the system of interacting one-dimensional maps the asymmetric component has a power-law probability density, with the power determined by the scaling function of the Lyapunov exponent. The high probability of large deviations is connected with the weak stability of the symmetrical state: if the coupling constant is increased, more and more trajectories lose stability, but below threshold the measure of unstable trajectories is zero.

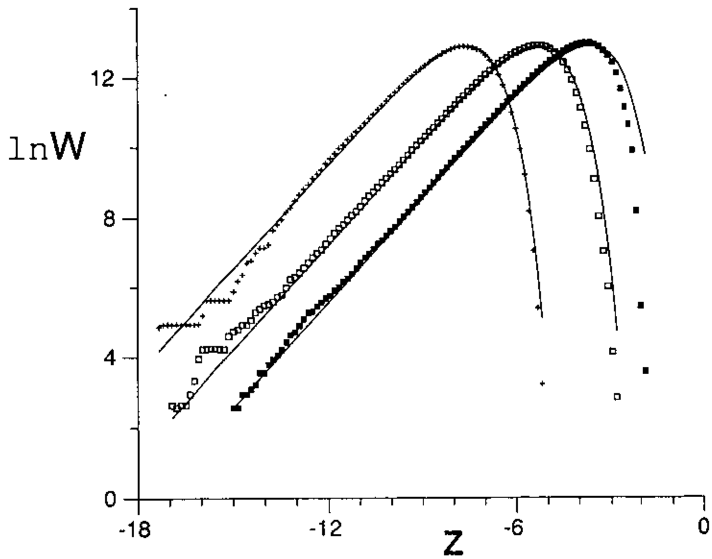

Figure 6. Numerically obtained probability density (in arbitrary units) in the system given by eqs. (1), (7) for $a = \frac{1}{2}$ and $\varepsilon = 10^{-3}$ (crosses), $\varepsilon = 10^{-2}$ (open squares), $\varepsilon = 5 \times 10^{-2}$ (filled squares). Comparison with the theoretical formula (32) shows better agreement for small ε.

Nevertheless, these trajectories have a very profound effect on the topology of the attractor basin, and imply that attractor definitions have to be chosen very carefully. Also, this instability is seen in perturbed systems, where the scaling law becomes manifest also in the phase with unbroken symmetry.

While we have treated a rather special system, we believe that similar results hold much more generally, namely whenever a strange attractor undergoes a continuous bifurcation. In such cases—which typically should be accompanied by a spontaneous symmetry breakdown just as conventional critical phenomena—some sets of points with measure zero will in general have anomalously large instability (like the fixed point in our example) or stability. These sets can be unstable (stable) though the system in globally stable (unstable). This will imply scaling laws and topological complications just as in our simple example. One practical consequence is that not perfectly clean systems will show 'precursors' of the bifurcation in the form of scale invariant fluctuations. They strongly enhance noise and small asymmetries, rendering thereby measurements of Lyapunov exponents by the method suggested in [8] very difficult, for example.

The essential part of our treatment is based on the fact that evolution of the asymmetrical component is governed by fluctuating local Lyapunov exponent. We expect that the same approach will be valid also in the other situations governed by fluctuating local expansion rates, for example in the statistics of trajectory separation in ensembles of systems subject to common noise [23].

Acknowledgments

One of the authors (AP) thanks the Alexander von Humboldt Stiftung for support. For very fruitful discussions we thank T von der Twer.

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