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Universal behaviour of two coupled circle maps

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Abstract. Symmetric coupling of two critical circle maps near the golden mean rotation number is considered. On the basis of a renormalization group method the three universal types of interaction are found. The theoretical scaling predictions are confirmed by numerical calculations.

1. Introduction

Three types of transition to chaos are now well studied: through period doubling [1], through intermittency [2] and through destruction of quasiperiodic motions [3, 4]. A common feature of these scenarios is the possibility of describing the critical phenomena by the renormalization group (RG) method. RG analysis allows one to find universal quantitative scaling laws both for a structure of a parameter space near a critical point and for the motions occurring. On the basis of the RG approach it is possible to describe the effect of external noise [5], the universal properties of response function [6], the scaling of multiple-frequency quasiperiodic motions [7], etc. An important generalization of the RG deals with continuous and coupled systems. For period doubling this generalization was constructed in [8], and for intermittency in [9]. Hamiltonian systems were considered in [10, 11]. In coupled systems universal types of interaction were found with non-trivial scaling properties.

In this paper we describe the universal types of interaction between critical circle mappings exhibiting transition to chaos through quasiperiodicity. For a golden mean rotation number we find three universal types of coupling. The RG method gives scaling constants which are confirmed numerically.

2. Renormalization group for a circle map

Transition to chaos through two-frequency torus destruction in dissipative systems may be correctly described by a circle mapping,

$$\vartheta_{i+1}=f(\vartheta_i)=\vartheta_i+\omega-\frac{a}{2\pi}\sin(2\pi\vartheta_i) \qquad (1)$$

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with two parameters. The parameter ω corresponds to the ratio of the frequencies of two linear oscillators, while the parameter a corresponds to the nonlinearity level. The mapping (1) is usually obtained as a phase transformation of one oscillator through a period of the second oscillator. For $a < 1$ the mapping (1) has either a stable periodic orbit (resonant state synchronization of the oscillators) or an ergodic invariant set (quasiperiodic state). Correspondingly, the rotation number $\rho = \lim_{n\to\infty}(f^n(\vartheta) - \vartheta)/n$ is either rational, $\rho = p/q$ (p and q are integers, and q is period of the orbit), or irrational. For $a > 1$ the mapping (1) is not one-to-one, so chaotic behaviour becomes possible. The transition from a periodic orbit to chaos was described in [4, 12, 13]. Inside a resonance region at the a, ω-parameter plane (figure 1) one can find the bifurcation lines: line of period doubling D; at lines A_L and A_R some iteration of the left (right) extremum of the mapping (1) coincides with the unstable periodic orbit. One can single out a point R with chaotic behaviour: for this parameter value both extrema are eventually periodic, so a mapping f^q has q-invariant intervals, and each transforms into itself with stretching. Quasiperiodic motion may be treated as a limit of adjusted resonances with large p and q, which approximate the irrational rotation number. For large q the point R approaches the critical line $a = 1$, so one may say that quasiperiodic motion breaks to chaos just at $a = 1$. For some irrational rotation numbers, approximating resonances are regularly scaled, suggesting the possibility of their RG treatment. These rotation numbers are represented by periodic continued fractions. We will consider, following [3, 4], the simplest rotation number—the so called golden mean $\rho = \sigma = (\sqrt{5} - 1)/2$, whose continued fraction is $(1, 1, 1, \ldots)$. Approximating fractions for σ are $\rho^{(n)} = F_{n-1}/F_n$, where F_n are the Fibbonachi numbers, which obey a recurrent relation $F_{n+1} = F_n + F_{n-1}$, $F_0 = F_1 = 1$. Correspondingly, there may be constructed a series of functions f_n obeying a renormalization relation [3],

$$f_{n+1} = f_n f_{n-1}. \tag{2}$$

After scaling of the variable ϑ with a constant α, the final transformation takes the form

$$f_{n+1}(\vartheta) = \alpha f_n\left(\alpha f_{n-1}\left(\frac{\vartheta}{\alpha^2}\right)\right) \qquad f_n(0) = 1. \tag{3}$$

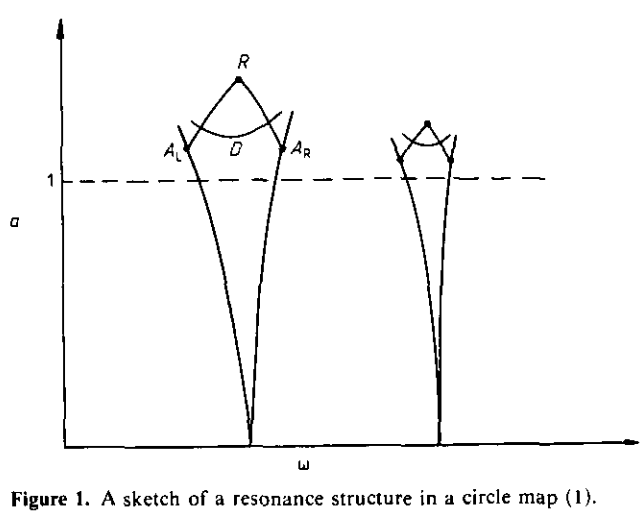

Figure 1. A sketch of a resonance structure in a circle map (1).

The transformation (3) has two fixed points: a trivial one, $\hat{f}(\vartheta) = -1 + \vartheta$, $\alpha = -\sigma^{-1}$, and a non-trivial analytic function of ϑ^3,

$$\hat{f}(\vartheta) = \sum_0^\infty C_m \vartheta^{3m} \tag{4}$$

found numerically in [3, 4]. The trivial fixed point describes quasiperiodic motions below the critical line ($a < 1$), while the non-trivial fixed point describes the neighbourhood of the critical point $a = 1$, $\omega = \omega_c$. The value of the constant $\alpha = -1.2886\ldots$ for the non-trivial fixed point gives scaling of the variable ϑ. An equation for perturbations near the RG fixed point has two significant eigenvalues with absolute values greater then 1: $\delta_1 = -2.8336\ldots$ and $\delta_2 = \alpha^2$. The constant δ_1 describes scaling along the critical line; in particular, for centres of resonances $\omega_0^{(n)}$ we have $(\omega_0^{(n)} - \omega_c) \propto \delta_1^{-n}$. The constant δ_2 describes scaling in the transverse direction. In particular, distances $\Delta a^{(n)}$ from the points R_n to the critical line scale as $\Delta a^{(n)} \propto \delta_2^{-n}$. Thus, the parameter plane (a, ω) scales in the vicinity of the critical point with constants δ_1 in ω and δ_2 in a.

3. Renormalization group for a coupled map

Let us consider a symmetrical interaction of two identical circle maps of type 1:

$$\begin{aligned} \vartheta_{i+1} &= f(\vartheta_i) + \varepsilon h(\vartheta_i, \varphi_i) \\ \varphi_{i+1} &= f(\varphi_i) + \varepsilon h(\varphi_i, \vartheta_i). \end{aligned} \tag{5}$$

Here ε is a small parameter-coupling constant. We will suppose that in synchronous mode, i.e. for $\vartheta = \varphi$, the interaction vanishes ($h(\vartheta, \vartheta) = 0$) and consider nearly synchronous states. Using the variables

$$u = \frac{\vartheta + \varphi}{2} \qquad v = \frac{\vartheta - \varphi}{2} \tag{6}$$

and neglecting $O(v^2)$ terms we obtain from (5)

$$u_{i+1} = f(u_i) \tag{7}$$

$$v_{i+1} = (f'(u_i) + \varepsilon \Psi(u_i)) v_i \tag{8}$$

where $\Psi(u) = \partial h(\vartheta, \varphi)/\partial\vartheta - \partial h(\vartheta, \varphi)/\partial\varphi|_{\vartheta=\varphi=u}$. Let us apply the renormalization transformation (2), (3) to equations (7) and (8). Denoting

$$\Phi(u) = f'(u) + \varepsilon \Psi(u)$$

we may write the renormalization transformation of (8) in the form

$$\Phi_{n+1}(u) = \Phi_n\left(\alpha f_{n-1}\left(\frac{u}{\alpha^2}\right)\right) \Phi_{n-1}\left(\frac{u}{\alpha^2}\right)$$

which in first order in ε gives

$$\Psi_{n+1} = \Psi_n\left(\alpha f_{n-1}\left(\frac{u}{\alpha^2}\right)\right) f'_{n-1}\left(\frac{u}{\alpha^2}\right) + \Psi_{n-1}\left(\frac{u}{\alpha^2}\right) f'_n\left(\alpha f_{n-1}\left(\frac{u}{\alpha^2}\right)\right).$$

Taking into account that $\lim_{n\to\infty} f_n = \hat{f}$ where $\hat{f}$ is the fixed point given by (4) (for the case of a trivial fixed point, see the appendix) we obtain finally

$$\Psi_{n+1} = \Psi_n\left(\alpha \hat{f}\left(\frac{u}{\alpha^2}\right)\right) \hat{f}'\left(\frac{u}{\alpha^2}\right) + \Psi_{n-1}\left(\frac{u}{\alpha^2}\right) \hat{f}'\left(\alpha \hat{f}\left(\frac{u}{\alpha^2}\right)\right). \tag{9}$$

We shall look for significant eigenvalues (with absolute values greater then 1) for the RG transformation (9) following [3, 4]. The eigenfunctions $\Psi(u)$ are supposed to be polynomials with minimal power k. After substitution $\Psi_n(u) \sim \lambda^n \Psi(u)$ in (9) we obtain from the coefficients at u^k for $k \neq 3l+2$,

$$\lambda^2 = \alpha^{-2k} \hat{f}'(\hat{f}(0)).$$

Taking into account that $\hat{f}'(\hat{f}(0)) = \alpha^4$ [3, 4] we find two significant eigenvalues,

$$\begin{aligned} \lambda_1 &= \alpha^2 = 1.66 \ldots \\ \lambda_3 &= \alpha = -1.28 \ldots. \end{aligned} \tag{10}$$

In the case $k=2$ the eigenfunction of (9) can be easily found analytically with the substitution $\Psi_n(u) = Q_n \hat{f}'(u)$. Then due to the identity

$$\hat{f}\left(\frac{u}{\alpha^2}\right) \hat{f}'\left(\alpha \hat{f}\left(\frac{u}{\alpha^2}\right)\right) = 1$$

we obtain

$$Q_{n+1} = Q_n + Q_{n-1}$$

so that

$$Q_n \propto (1+\sigma)^n.$$

The third eigenvalue is therefore

$$\lambda_2 = 1 + \sigma = 1.61 \ldots. \tag{11}$$

Thus, there are three significant eigenvalues, λ_1, λ_2 and λ_3, and, correspondingly, three non-trivial types of interaction. The scaling laws are considered below.

4. Numerical analysis of scaling

The RG analysis given in sections 3 and 4 predicts that in a coupled circle map one may observe scaling in a five-parameter space (two parameters correspond to an uncoupled map and three correspond to significant types of interaction). However, in order to obtain appreciable results we shall consider scaling in two-parameter subspaces separately for different coupling modes.

(*a*) The coupling of the first type occurs if $\Psi(0) \neq 0$. This coupling may be represented by the following system:

$$\begin{aligned} \vartheta_{i+1} &= \vartheta_i + \omega - \frac{a}{2\pi} \sin(2\pi\vartheta_i) + \frac{\varepsilon_1}{4\pi} \sin[2\pi(\vartheta_i - \varphi_i)] \\ \varphi_{i+1} &= \varphi_i + \omega - \frac{a}{2\pi} \sin(2\pi\varphi_i) + \frac{\varepsilon_1}{4\pi} \sin[2\pi(\varphi_i - \vartheta_i)]. \end{aligned} \tag{12}$$

We shall fix the critical parameter value $a=1$ and consider the bifurcation structure in the plane (ε_1, ω). The synchronous regimes $\vartheta_i = \varphi_i$ for resonances $\rho^{(n)}$ are stable inside the shaded regions in figure 2. One can see that the whole diagram scales if the parameters ω and ε_1 are multiplied simultaneously by δ_1 and λ_1. For fixed ω inside

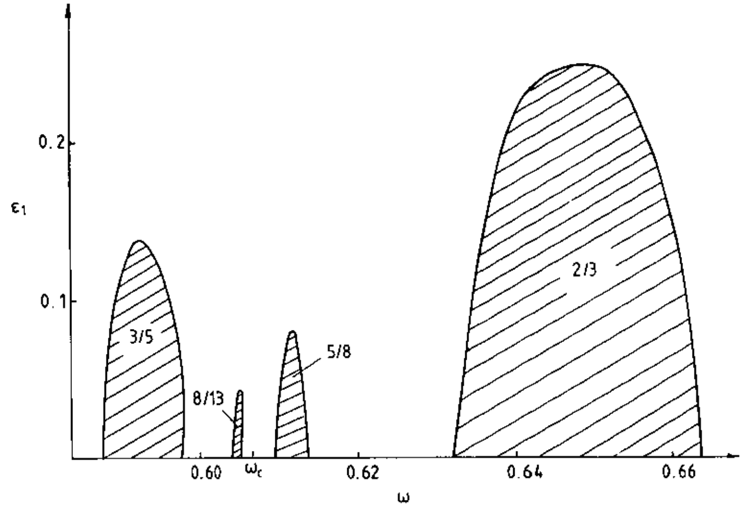

Figure 2. Stability regions for resonances of type-1 coupling.

a resonance the synchronous state loses its stability as ε_1 increases with the appearance of a non-symmetric cycle.

(b) For the second type of coupling we know the eigenfunction exactly: $\Psi(u)=\hat{f}'(u)$. The corresponding coupled map system is as follows:

$$\vartheta_{i+1}=f(\vartheta_i)+\frac{\varepsilon_2}{2}(f(\varphi_i)-f(\vartheta_i))$$
$$\varphi_{i+1}=f(\varphi_i)+\frac{\varepsilon_2}{2}(f(\vartheta_i)-f(\varphi_i)). \tag{13}$$

This type of interaction was investigated in [8, 14] where it was called dissipative interaction. Using the variables (6) for small v we get

$$v_{i+1}=(1-\varepsilon_2)f'(u_i)v_i. \tag{14}$$

It is easily seen from (14) that dissipative coupling with positive ε_2 leads to the decrease of the variable v. Thus the coupling tends to synchronize the interacting systems. A non-trivial regime may be observed if the synchronous state is chaotic, i.e. the Lyapunov number $\Lambda=\langle\ln|f'(u_i)|\rangle$ is positive. In this case there exists a critical value ε_{2c} for which a slightly inhomogeneous chaotic regime sets in (see figure 3). The critical value is easily obtained from (14):

$$\ln(1-\varepsilon_{2c})+\Lambda=0. \tag{15}$$

The chaotic states of figure 3 correspond to the points R_n in the plane (a,ω) (see figure 1). For the points R_n we have $\Lambda_n\sim F_n^{-1}\sim(1+\sigma)^{-n}$, thus from (15) we obtain $\varepsilon_{2c}\sim(1+\sigma)^{-n}$ in accordance with the scaling law (11).

(c) For the type-3 coupling $\Psi(u)\sim u$ but we do not know the eigenfunction $\Psi(u)$ exactly. However, it is necessary to know the coupling term $h(\vartheta,\varphi)$ accurately because a small part of type-1 or type-2 coupling will disturb the scaling properties due to the

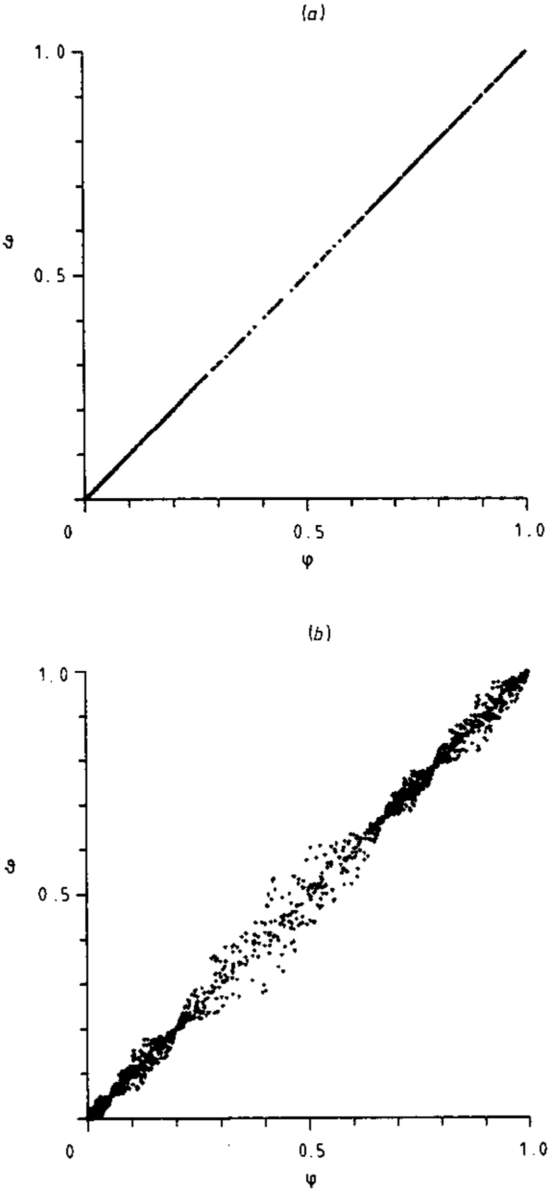

Figure 3. Effect of type-2 coupling on the chaotic motion inside the resonance 3/5: (*a*) $\varepsilon_2 > \varepsilon_{2cr}$; (*b*) $\varepsilon_2 < \varepsilon_{2cr}$.

relations $\lambda_3 < \lambda_1$, $\lambda_3 < \lambda_2$. Numerical experiments showed that with good accuracy 'pure' type-3 coupling may be observed in the following system:

$$
\begin{aligned}
\vartheta_{i+1} &= f(\vartheta_i) + \frac{\varepsilon_3}{4\pi}(\cos 2\pi\vartheta_i + 0.565\cos 4\pi\vartheta_i - \cos 2\pi\varphi_i - 0.565\cos 4\pi\varphi_i) \\
\varphi_{i+1} &= f(\varphi_i) + \frac{\varepsilon_3}{4\pi}(\cos 2\pi\varphi_i + 0.565\cos 4\pi\varphi_i - \cos 2\pi\vartheta_i - 0.565\cos 4\pi\vartheta_i).
\end{aligned}
\qquad (16)
$$

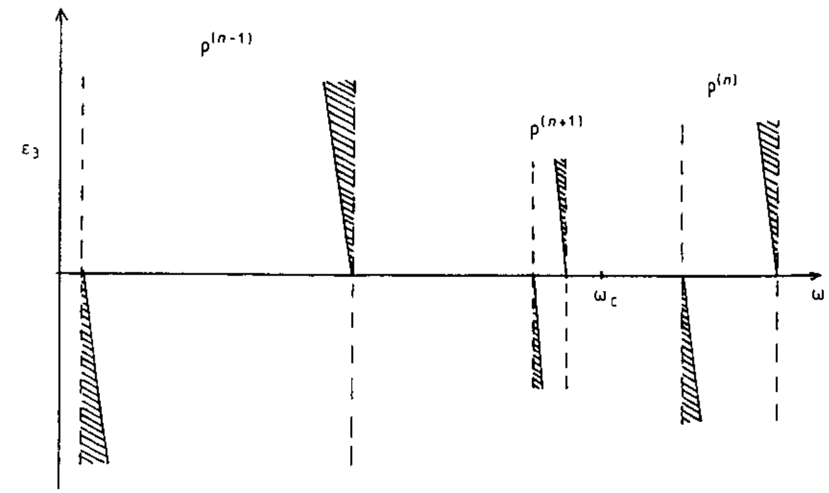

Figure 4. A sketch of the stability structure for $\rho^{(n)}$-resonances of type-3 coupling. The instability regions are shaded.

Similarly to type-1 coupling we constructed the regions of stability of synchronous states in the plane (ε_3, ω) (figure 4). Here, because λ_3 is negative, the instability may occur for both signs of ε_3. However, the instability regions are very small and adjusted to ends of phase-locking intervals. The whole diagram scales if one multiplies $(\omega - \omega_c)$ and ε_3 by δ_1 and λ_3.

5. Conclusion

We considered the interaction of two maps exhibiting the transition to chaos through the golden mean quasiperiodic state. It was shown that there are three significant types of coupling (one dissipative and two inertial) with the scaling constants (10) and (11). Bifurcation diagrams demonstrate scaling properties of adjusted resonances. The type-1 coupling with the largest eigenvalue is most significant. This coupling leads to the system desynchronization and to onset of asynchronous periodic and quasiperiodic regimes. Description of the full bifurcation diagram is, however, beyond the scope of the present paper.

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Appendix

For a subcritical case the RG transformation (3) has a fixed point

$$\hat{f}(\vartheta) = -1 + \vartheta \qquad \alpha = -\sigma^{-1}.$$

We will use a method described in [4] for obtaining the eigenfunctions of the RG transformation (9). Denoting $\Psi_n(u)=\lambda_m^n\tilde{\Psi}_m(u)$, where m corresponds to a maximal degree of the polinomial $\tilde{\Psi}_m(u)$, it follows from (9) that

$$\lambda_m^{n+1}\tilde{\Psi}_m(u)=\lambda_m^n\tilde{\Psi}_m\left(-\alpha+\frac{u}{\alpha}\right)+\lambda_m^{n-1}\tilde{\Psi}_m\left(\frac{u}{\alpha}\right).$$

For the maximal degree u^m we obtain

$$\lambda_m=-\alpha^{1-m} \qquad \text{or} \qquad \lambda_m=\alpha^{-1-m}.$$

There is only one significant eigenvalue $\lambda_0=-\alpha$ and the corresponding eigenfunction is $\hat{\Psi}_0\sim\hat{f}'$. When compared with the results of sections 3 and 4 it is evident that this coupling is of a dissipative type.

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