SPACE–TIME DEVELOPMENT OF CHAOS IN NONEQUILIBRIUM MEDIA

A. S. Pikovskii

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Using a space–time analogy, we consider the time development of spatial chaos in an infinite medium and the spatial development of timelike chaos in a semi-infinite medium. Landau–Ginzburg equations are used to describe the secondary instability periodic in one of the field coordinates. It is shown that an increase in the combinational frequencies leads to a dense spectrum that is practically indistinguishable from a continuous one; in phase space, this corresponds to the formation of creases on the two-dimensional torus.

1. There has been a great deal of interest recently in the study of stochasticity in extended systems. Here regimes are possible that are similar to those observed in lumped systems. For example, the field in a resonator can be represented as a set of discrete modes, whose time evolution is described by a strange attractor. The situation changes if one considers a spatially infinite medium. In this case, the spatial spectrum may be continuous, and the problem arises of the development of spatial chaos. In general, in extended systems there is a great variety of ways of setting up the problems, associated with choosing the initial and boundary conditions. In this paper, we consider the situation where one is given the dependence of the field on one variable and then follows the evolution in the other variable. In terms of a space-time analogy, we are here dealing with two types of problems: 1) in an infinite medium, the distribution of the field at the initial time is given with respect to the coordinate. Then, one follows the evolution of this field in time, in particular, the onset and development in time of spatial chaos; 2) in a semiinfinite medium the dependence of the field on time is given on the boundary. One follows the onset and spatial development of timelike chaos. We note that some regimes of the spatial development of chaos have been discussed in [1-3].

2. The problem of the development of timelike chaos in space will be discussed using the example of the complex Landau–Ginzburg equation

$$\frac{\partial a}{\partial x} = a + (1+ic_1)\frac{\partial^2 a}{\partial t^2} + (-1+ic_2)|a|^2 a. \qquad (1)$$

This equation is the basic equation of a model describing quasiharmonic waves in a nonequailibrium medium having convective instability. In fact, let linear waves of the form exp(ikx–iωt) satisfy the dispersion relation D(ω, k; r) = 0 (r is a parameter corresponding to the degree of nonequilibrium). This equation determines the complex wave number k as a function of the real frequency ω. Near the minimum of the neutral curve, on which the spatial increment Im k goes to zero (the point k_c, ω_c, r_c), we have

$$k = k_c + \frac{\partial k}{\partial \omega}(\omega-\omega_c) + \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}(\omega-\omega_c)^2 + \frac{\partial k}{\partial \omega}(r-r_c) + \ldots$$

This representation of the dispersion relation corresponds to a weakly-nonlinear partial differential equation for the slowly varying complex amplitude A(x, t)

$$i\frac{\partial A}{\partial x} = i\frac{\partial k}{\partial \omega}\frac{\partial A}{\partial t} - \frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\frac{\partial^2 A}{\partial x^2} + \frac{\partial k}{\partial r}(r-r_c)A + d|A|^2 A. \qquad (2)$$

We remark that the quantities $\partial^2 k/\partial\omega^2$ and d are complex. By means of a change of variables, (2) can be brought to the form (1).

For Eq. (1) the problem can be posed as follows: at x = 0 the field a(0, t) on the boundary of the medium is given, $-\infty < t < \infty$, and we want to find the field for x > 0. We

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emphasize that this way of posing the problem differs significantly from the usual statement of the chaos problem in the theory of dynamical systems (see, for example, [2]). Usually one is given the initial conditions and then looks at the evolution in time. In a medium having convective instability, an initial perturbation tends to "run away," and a nontrivial regime is observed only if in some region or at some point there is a continually acting perturbation. Thus, we are led to the problem of transforming the boundary conditions. It then makes sense to speak of the onset of chaos if the perturbations on the boundary are regular, i.e., are periodic or quasiperiodic. If the perturbations on the boundary have a noiselike, fluctuating behavior, then we are led to the relatively trivial problem of the transformation of noise.

Let us consider the case of a periodic perturbation on the boundary $a(0, t) = a(0, t+T)$. Then, in view of the invariance of Eq. (1) with respect to shifts in t, we obtain $a(x, t) = a(x, t + T)$, and, accordingly, at all points in space we will observe a regime periodic in time. Then, the variation of the field as a function of the coordinate may be chaotic (in particular, such regimes were obtained numerically in [4-6]). Let us consider the instability of such a regime of $a(x, t)$, periodic in t.

For a small perturbation $b(x, t)$, we have, after linearizing (1),

$$\frac{\partial b}{\partial x} = b+(1+iC_1)\frac{\partial^2 b}{\partial t^2} + (-1+iC_2)(a^2b^* + 2|a|^2 b). \tag{3}$$

In the linear equation (3), the coefficients of a^2 and $|a|^2$ are periodic functions of t with period T, and their variation with respect to x may be chaotic. Since the time period of the secondary perturbation $b(x, t)$ is not necessarily equal to T, we will look for the fundamental solution of (3) in the form

$$b(x,t) = \exp(i\nu t)u_1(x,t)+\exp(-i\nu t)u_2(x,t), \tag{4}$$

where $u_{1,2}(x, t) = u_{1,2}(x, t + T)$ are periodic functions of t, and the parameter ν is the "quasifrequency" of the secondary perturbation. We note that the substitution (4) is analogous to the Bloch substitution for finding the eigenfunctions of a periodic potential. As a result, for $u_{1,2}$ we obtain the system of linear equations

$$\frac{\partial u_{1,2}}{\partial x} = u_{1,2}+(1+iC_1)\left(\frac{\partial^2 u_{1,2}}{\partial t^2} \pm 2i\nu\frac{\partial u_{1,2}}{\partial t} - \nu^2 u_{1,2}\right) + (-1+iC_2)(a^2u^*_{2,1}+2|a|^2 u_{1,2}). \tag{5}$$

The solution of the system (5) behaves for large x as $\exp(\lambda x)$. The exponent $\lambda(\nu)$, which determines the stability of the secondary perturbation, we will refer to as quasi-Liapunov, since it depends on the quasifrequency ν, and at $\nu = 0$ goes over to the usual Liapunov exponent. For a regime periodic in t $\lambda(0) = 0$, and for a chaotic regime $\lambda(0) > 0$.

The quasi-Liapunov exponent can be calculated exactly by the same method as the usual Liapunov exponent [7]. Figure 1 shows the result of calculating the stability of the evolution of a field periodic in t on the boundary, for the case $C_1 = 3$, $C_2 = 5$ and various

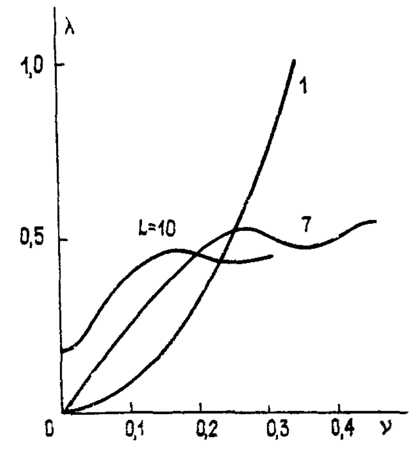

Fig. 1

values of the period T. When T = 1 in (1), a regime is established where the amplitude is constant in x, and when T = 7 the amplitude of the perturbation varies periodically in x, and when T = 10 it varies chaotically. In all of these regimes there is a secondary instability, indicating the growth with x of secondary perturbations having periods not coinciding with T. Thus, although a perturbation periodic on the boundary remains periodic at all points in space, a secondary instability may develop which breaks this periodicity. The detailed behavior of the process depends on the specific form of the secondary perturbations. If the secondary perturbations are fluctuating, then an increase may be observed of the noise components in space, which for large x goes over to space-time turbulence. A more complicated picture may arise if the secondary perturbations are periodic, but their period is incommensurate with T. Let the field on the boundary have the form $a(0, t) = a_0(t) + \varepsilon a_1(t)$, where a_0 is a periodic function with period T_0, and a_1 has period T_1, such that the frequencies $\omega_0 = 2\pi/T_0$ and $\omega_1 = 2\pi/T_1$ are incommensurate, $\varepsilon \ll 1$. Qualitatively, the development of the process in space may occur in the following way. For small x a regime chaotic in x develops, having time period T. Then, in view of the above-described secondary instability, the perturbation $a_1(x, t)$ starts to grow. As it develops, due to the nonlinearity there arise all possible combinational frequencies $n\omega_0 + m\omega_1$, which also enter the region of secondary instability and start to increase. As a result, for large x the spectrum approaches a continuous one, while remaining in a strict sense discrete. This process was modeled by us numerically. Equation (1) was solved on a grid using an implicit differencing scheme. Since it is not possible to solve the equations in the region of infinite t, we used periodic conditions in t. Then, the incommensurate frequencies become commensurate: $\omega_0/\omega_1 = p/q$,

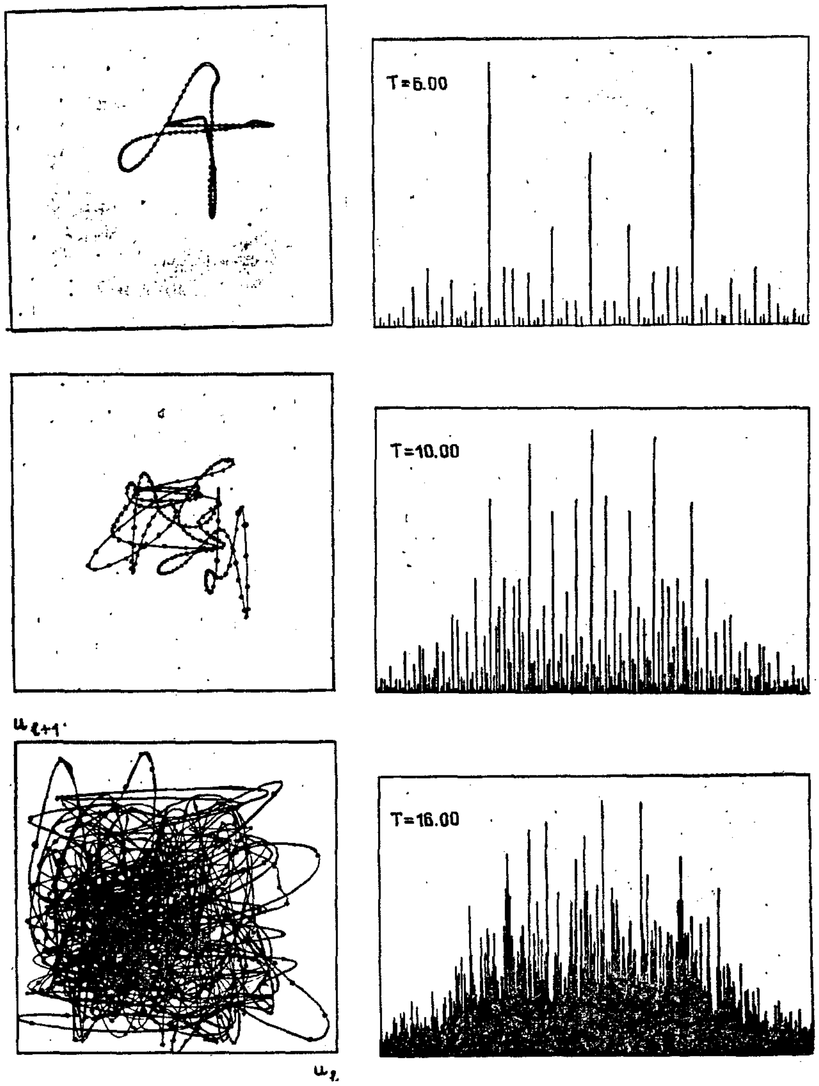

Fig. 2

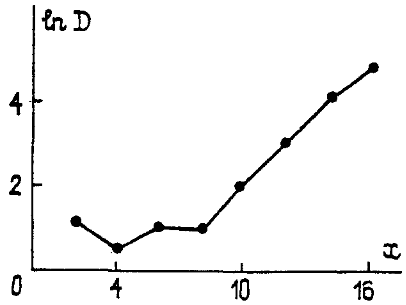

Fig. 3

but for large p and q this may be a good approximation to some irrational number, and a process may be modeled which occurs for quasiperiodic perturbations on the boundary. As an example, the irrational number $(\sqrt{5} - 1)/2$ is approximated by the ratio $p/q = 377/610$. The development in x of a timelike perturbation spectrum, having frequencies $\omega_0 = \pi/10$, $\omega_1 = \omega_0 \cdot 377/610$, is shown in Fig. 2. It is seen that as T increases the spectrum gets filled in, which may be interpreted as a turbulent process. Here a paradox arises: the process, strictly speaking, is regular (quasiperiodic), but the spectrum appears to be continuous. To resolve this paradox, let us consider the evolution of the quasiperiodic regime not from the point of view of the spectrum, but from the point of view of phase space: we form the sequence of real numbers $u_l = |a(x, lT_0)|^2$, $l = 0, 1, 2, \ldots$ and map them into phase space $(u_\ell, u_{\ell+1})$. When $0 < \ell < \infty$, such points fill in a closed line, being the cross section of a quasiperiodic winding of a two-dimensional torus. In our numerical calculations, we obtained a set of q points approximating this line. Figure 2 shows the results of such an analysis for various x. It is seen that as x increase the torus becomes wrinkled, and the length of its cross section increases, such that this increase, as follows from Fig. 3, is exponential. For large x, the tangled, interwoven line is practically indistinguishable from the projection of a multidimensional strange attractor onto a plane. Thus, in phase space, as in the spectrum, we have what is strictly speaking a regular object – a wrinkled torus, which for large x looks like a strange attractor. The spatial development of timelike chaos in the system (1) is also manifested as just such an evolution of the torus.

3. The problem described above can be formulated as the development in space of timelike chaos. Using a space-time analogy, we can apply our approach to describing the time variation of the spatial structure of the field. In fact, to describe this process, we can directly use the results of Section 2. For example, for quasiharmonic waves we can, analogously to what was done above, but with the change $x \leftrightarrow t$, $\omega \leftrightarrow k$, obtain the equation

$$\frac{\partial a}{\partial t} = a + (1 + i\bar{C}_1)\frac{\partial^2 a}{\partial x^2} + (-1 + i\bar{C}_2)|a|^2 a, \tag{6}$$

describing the time evolution of the initial field a(x, 0). An initally periodic field remains periodic, but there may occur a secondary instability having other wave numbers. A quasiperiodic initial field becomes chaotic in precisely the same way as described in Sec. 2, the spatial spectrum approaches a continuous one, and a wrinkled torus is formed in the effective phase space. We emphasize that we may speak of a spatial distribution becoming periodic only in an infinite medium, since in the presence of boundaries the spatial spectrum is always discrete.

This phenomenon of the development of space-time chaos is observed not only in the Ginzburg–Landau equation, but also in a number of other systems. For example, one-dimensional waves on a running-off thin film of liquid are described by the Kuramo–Sivashinskii equation [8-10]

$$\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial t^4} = 0\,. \tag{7}$$

For this equation, we can pose the problem of the development of timelike chaos in space. For a quasiperiodic perturbation of the waves on the boundary x = 0, a filling-in of the spectrum is observed, analogous to Fig. 2. Above we discussed only systems continuous in space and in time. Our approach can also be extended to discrete (in one or both variables) models. For example, the development in time of spatial chaos may be observed in the conservative chain

$$\frac{d^2u_i}{dt^2} - (u_{i-1} - 2u_i + u_{i+1}) - u_i + u_i^3 = 0, \quad -\infty < i < \infty. \tag{8}$$

Here the quasiperiodic initial condition is given, for example, by the function $u_i = \varepsilon\cos(2\pi\rho i)$, where ρ is an irrational number, and in the process of evolution such a distribution becomes chaotic. We remark that for chains like (8), the effect of secondary instability has been observed even in early numerical experiments [11]. In these experiments periodic chains $1 < i < N$ were studied, the initial field had period N/2, and as a result of the evolution due to the secondary instability there occurred a "spontaneous" breaking of the period N/2.

Another discrete model, in which one can observe the development of timelike chaos in space, is a chain of nonlinear amplifiers [12]. In the simplest case, it is described by the system of equations

$$\frac{du_i}{dt} + u_i = f(u_{i-1}), \quad i = 1, 2, 3 \ldots, \tag{9}$$

where f(u) is the nonlinear response function of the amplifier. With an input signal $u_0(t)$ in the form of a quasiperiodic function, it becomes chaotic along the "spatial coordinate" i. We note that in this situation the wrinkled torus arises in the natural phase space of a system of ordinary differential equations (9).

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