TRANSITION FROM A SYMMETRIC TO A NONSYMMETRIC REGIME UNDER CONDITIONS OF RANDOMNESS DYNAMICS IN A SYSTEM OF DISSIPATIVELY COUPLED RECURRENCE MAPPINGS

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We undertake a study on a model system of the transition from a cophasal to a non-cophasal regime in the dynamics of randomness as the magnitude of the dissipative coupling is reduced. Expressions are derived for the distribution density and for the moment of the dynamic variables near the point of transition from one disordered random state to another; agreement with results from numerical experimentation is demonstrated.

A considerable amount of information is now at hand with regard to the dynamic randomness that exists in simple nonlinear oscillating systems so that it now becomes natural to turn to more complex problems in which these systems serve as elementary building blocks. In the present work, along the lines of this program, we study a system of two identically coupled subsystems, each of which demonstrates a random nature. We will assume that the variables u and v describe the states of the two subsystems, while the parameter γ defines the magnitude of the coupling between them.

As regards the nature of the coupling, we will assume that it is dissipative, i.e., it enhances the evening out of the instantaneous states of the subsystems, and when the instantaneous states are equal it has no effect on their dynamics. The nature of the regime observed within the system is determined through competition between two factors. The first of these factors involves the scattering of the phase trajectories, caused by the random dynamics of the subsystems: if the initial states are easily distinguished, the scattering of the trajectories will serve to enhance this distinction. The second factor is the effect of the dissipative coupling which, as was stated earlier, enhances the evening out of the states. In the case of a large coupling, it is the second factor that predominates and a cophasal random regime arises within the system. Here, in the plane of variables u and v the representational point executes random oscillations, remaining always on the bisectrix (Fig. 1a). With a reduction in the coupling parameter the first factor begins to predominate at some instant of time, i.e., the scattering of the trajectories, and the random state becomes non-cophasal: the representational point on the u, v plane no longer remains on the bisectrix, but executes motion in its vicinity (Fig. 1b). In the place of the coupling parameter γ it will become convenient for us to use the quantity ε which characterizes the supercritical antisymmetric components of the solution: $\varepsilon < 0$ corresponds to a cophasal random state, $\varepsilon > 0$ corresponds to a non-cophasal random state, and $\varepsilon = 0$ corresponds to the critical situation (to the instant of transition). The problem thus calls for an examination of the onset of non-cophasal randomness on passage of ε through zero, relying, where possible, exclusively on the smallness of ε and avoiding any additional unnatural assumptions.

1. Model. Let us examine the model system of two coupled one-dimensional recurrence mappings of the following form:

$$u_{n+1}=f(u_n)+\gamma[f(v_n)-f(u_n)], \qquad v_{n+1}=f(v_n)+\gamma[f(u_n)-f(v_n)], \tag{1}$$

where n is discrete time, u_n and v_n are the states of the subsystems at the instant n, $\gamma \in [0, 1]$ is the coefficient of dissipative coupling. We will give the function f in the form

$$f(u) = 1 - 2u^2, \tag{2}$$

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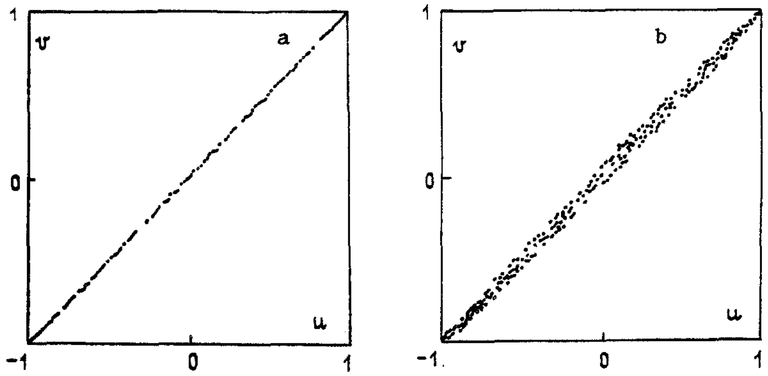

Fig. 1

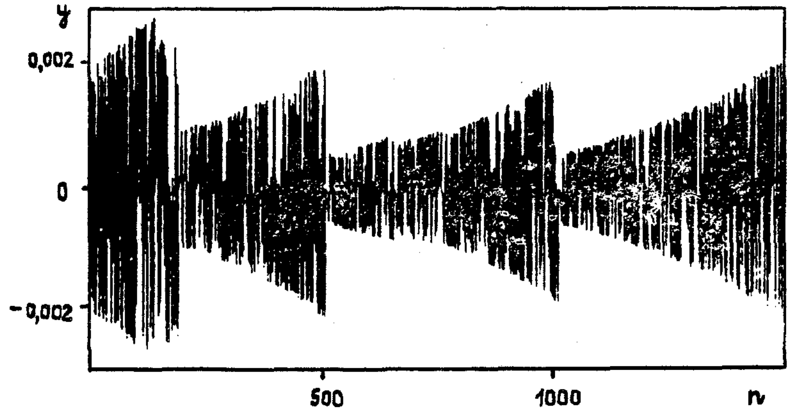

Fig. 2

and in this case the dynamics of the individual mapping $u_{n+1} = f(u_n)$ is, as is well known, random in nature [1].

If the initial values of the dynamic variables u and v lie in the interval [−1, 1], then with the indicated method of introducing the couplings they remain bounded for all of the subsequent instants of discrete time. Indeed, when $|u_n| \leq 1$, $|v_n| \leq 1$, it follows from (2) that $|f(u_n)| \leq 1$, $|f(v_n)| \leq 1$, so that according to (1), $|u_{n+1}| \leq (1-\gamma)|f(u_n)| + \gamma|f(v_n)| \leq 1$ and, analogously, $|v_{n+1}| \leq 1$.

In the place of u and v let us introduce the new variables x and y, which, respectively, characterize the symmetric and antisymmetric parts of the solution:

$$x = 1/2\ (u+v), \quad y = 1/2(u-v)\ , \tag{3}$$

and we will assume $\gamma = (1-\varepsilon)/4$. (As can be demonstrated, the critical situation for the onset of the non-cophasal randomness with the chosen function f(u) corresponds to $\gamma = 1/4$ [2].) System (1) is then rewritten in the form

$$x_{n+1} = 1-2x_n^2-2y_n^2, \quad y_{n+1} = -2(1+\varepsilon)x_n y_n\,. \tag{4}$$

A numerical investigation into the dynamics of such a system was undertaken by the Japanese authors of [3]. Figure 2 reproduces the relationship which they obtained between the antisymmetric portion y_n of the solution and the discrete time n for low supercriticality $\varepsilon = 0.003$. We see that the solution is a succession of growing trains whose duration varies randomly from time to time. Each train ends with a sharp drop in the amplitude of the non-cophasal component to some small level, subsequent to which a new growth cycle begins. This developing theory must necessarily serve to explain the unique features of this process.

2. Theory. In zeroth approximation, considering the supercriticality of ε and the antisymmetric component y to be infinitely small, from Eq. (4) we obtain

$$x_{n+1} = 1-2x_n^2, \quad y_{n+1} = -2x_n y_n\,. \tag{5}$$

Equations (5) have a "first integral" $|y_n| = C\sqrt{1 - x_n^2}$, where C is some arbitrary constant, which can be proved through direct substitution of this expression into (5). This observation is of fundamental meaning for subsequent analysis. It follows therefrom that it is expedient to replace x_n and y_n by a new pair of dynamic variables x_n and C_n, where

$$C_n = |y_n|/\sqrt{1-x_n^2}. \tag{6}$$

The level of the antisymmetric component in this case is characterized by C_n, which will change slowly over time, whereas y_n contains a fast component that is associated with the random oscillations of the variable x_n.

Equation (4) is rewritten in the new variables as follows:

$$x_{n+1} = 1 - 2x_n^2 - 2C_n^2(1 - x_n^2),$$
$$C_{n+1} = C_n \frac{1+\varepsilon}{\sqrt{[1 + (x_n^{-2} - 1) C_n^2](1 - C_n^2)}}. \tag{7}$$

Limiting ourselves to a situation close to the critical, in which $\varepsilon \ll 1$ and $C_n \ll 1$, we will neglect all of the terms with the exception of the nonvanishing first-order terms of ε and C, and from the second equation in (7) we will obtain

$$C_{n+1} = C_n \frac{1+\varepsilon}{\sqrt{1 + \eta_n C_n^2}}, \quad \eta_n = \frac{1}{x_n^2}. \tag{8}$$

This relationship may be regarded as an equation for the determination of C_n, which contains the random coefficient η_n. With small ε the statistic of the random quantity η_n may, apparently, be regarded as independent of C_n and it can be found, by using the unperturbed equation $x_{n+1} = 1 - 2x_n^2$ for x_n. Formula (8) makes clear the specific nature that the action of the nonlinearity of the system exerts on the dynamics of the antisymmetric component: the nonlinear term is small because of the smallness of C_n to the point at which the variable x_n is no longer close to zero. At this instant η_n becomes large, the role of nonlinearity increases sharply, and there is a marked reduction in the level of the antisymmetric component. It is precisely this that constitutes the mechanism of detachment of the growing oscillation trains observed in the numerical calculations in [3].

With replacement of $Z_n = 1/C_n^2$, Eq. (8) reduces to a linear equation with an additive random factor

$$Z_{n+1} = (1-2\varepsilon) Z_n + \eta_n. \tag{9}$$

Some complexity is introduced by the fact that the random quantity η_n includes no finite moments, and Eq. (9) is therefore not solved by the traditional Fokker–Planck method.

For the solution of the problem we will make use of the method of characteristic functions. We will calculate the characteristic function for the quantity η_n by using the familiar relationship for the invariant distribution of the quantity x in the form $1/\pi\sqrt{1 - x^2}$ [1]. The corresponding integral is presented in the form

$$\varphi_\eta(w) = \langle e^{iw\eta_n}\rangle = \frac{1}{\pi}\int_{-1}^{1} \frac{e^{iw/x^2}}{\sqrt{1-x^2}} dx = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}, iw\right) = 1 - 2\sqrt{\frac{|w|}{\pi i}} + O(w^{3/2}), \tag{10}$$

where Γ is an incomplete gamma function.

The random pulses in Eq. (9) for the asymptotic case of small ε under consideration may be regarded as statistically independent because significant high-level pulses occur quite rarely, so that because of the random nature in the dynamics of the variable x_n the statistical coupling between them disappears. Bearing this circumstance in mind, we derive the following equation for the characteristic function of the quantity Z:

$$\varphi_{n+1}(w) = \varphi_n(w(1-2\varepsilon))\varphi_\eta(w), \quad \varphi_n(w) = \langle e^{iwZ_n}\rangle. \tag{11}$$

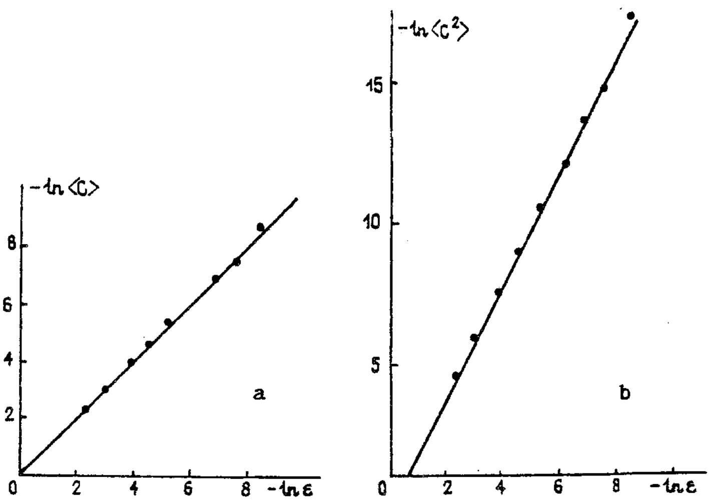

Fig. 3

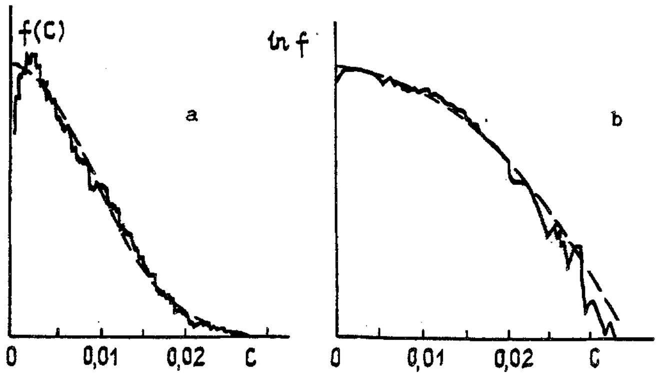

Fig. 4

The stationary solution of this equation has the form

$$\varphi(w) = \prod_{n=0}^{\infty} \varphi_\eta((1-2\varepsilon)^n w) . \tag{12}$$

In approximation of (10) we obtain

$$\varphi(w) = \exp\left(-\frac{2}{\varepsilon}\sqrt{\frac{|w|}{\pi i}}\right), \tag{13}$$

from which, by means of an inverse Fourier transform, we find the distribution density for the quantity Z, the so-called Levi distribution [4]:

$$F(Z) = \frac{1}{\pi\varepsilon Z^{3/2}}\, e^{-1/\pi\varepsilon^2 Z} , \tag{14}$$

and from (14) we obtain the distribution density for the quantity C, characterizing the level of the antisymmetric component:

$$F(C) = \begin{cases} \frac{2}{\pi\varepsilon}\, e^{-C^2/\pi\varepsilon^2}, & C > 0 \\ 0, & C < 0 \end{cases} . \tag{15}$$

Let us note that the function F depends on the combination C/ε, from which follows the presence of scaling: as the supercriticality of ε is reduced severalfold, the characteristic amplitude of the non-cophasal component is reduced in equal measure, while the distribution function remains similar in form to the original. Using (15), we obtain the following relationships for the C moments:

$$\langle C\rangle = \varepsilon, \quad \langle C^2\rangle = \pi\varepsilon^2/2. \tag{16}$$

Since the quantities x and C exhibit totally diverse characteristics of variation in time (for x on the order of unity, while for C on the order of ε^{-1}), they can be regarded as independent and we can write the combined distribution density in the form of the product

$$F(x, C) = \frac{2}{\pi\varepsilon} \exp\left(-C^2/\pi\varepsilon^2\right) \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}.$$

Returning to the variables x and y, for the two-dimensional density we obtain the expression

$$F(x, y) = \frac{1}{\pi\varepsilon}(1-x^2)^{-1} \exp\left(-\frac{1}{\pi\varepsilon^2} \frac{y^2}{1-x^2}\right). \tag{17}$$

At the limit $\varepsilon \to 0$ from (17) we obtain the distribution at the segment [−1, 1] of the x axis or, in terms of the original variables u and v, on the bisectrix u = v. With increasing ε the width of the distribution in the y direction increases proportionally to ε. From (17) we can find the dispersion of y for a fixed x which behaves as $\langle y^2\rangle = (\pi\varepsilon^2/2)\sqrt{1-x^2}$. In qualitative agreement with Fig. 1b the dispersion attains a maximum in the middle of the x-variation interval and drops off to zero at the edges.

3. Numerical Experiments. To verify the derived results, we carried out numerical experiments during which repeated iterations of the two-dimensional mappings (4) were performed for small values of ε and we calculated the statistical characteristics of $C = |y|/\sqrt{1-x^2}$.

Figure 3 shows a comparison of the relationship between the first and second moments $\langle C\rangle$ and $\langle C^2\rangle$ and the supercriticality of ε. The dots indicate results from the numerical experiment, while the solid lines represent the theoretical relationships (16). Figure 4 shows a comparison between the empirical and theoretical distribution density for the quantity C when $\varepsilon = 0.01$: the solid line corresponds to the numerical experiment, while the dashed line corresponds to the theoretical formula (15). The resulting agreement must be regarded as quite good. Some divergence appears in the shape of the empirical and theoretical distribution functions. Processing of these data with respect to the criterion χ^2 demonstrates that this divergence is not statistical but systematic in nature, i.e., it is obviously associated with the approximations made during the theoretical analysis. It might be assumed that the divergence will diminish with a reduction in ε.

The specific system under consideration exhibits two significant unique features. First of all, the invariant distribution density for an individual subsystem is calculated explicitly; secondly, the randomness in the individual subsystem is characterized by a zero nonuniformity factor (the characteristic of strange attractors, introduced in [5, 6]). For this reason, these results cannot lay claim to universality. Nevertheless, in our opinion, they are of some interest as a first example of an analytical solution for the problem of the transition from randomness to randomness and might serve as the starting point for further development of the theory of such transitions.

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