UDC 621.385.6.01

Randomization of Electromagnetic Radiation in Systems with Convective Instability in the Electron Beam*

N. S. GINZBURG, A. S. PIKOVSKII, AND A. S. SERGEYEV

A space-time approach is developed to analyze the evolution of narrow-band signals in systems with convective instability in the electron beam (plasma-beam systems, and microwave electron devices). Using a model with a beam of ideally bunched electrons as a framework, we show that when the beam interacts with a monochromatic wave, this renders the monochromatic wave unstable to the excitation of satellite waves with frequencies different from that of the main wave. In the nonlinear stage of its development, this instability leads to a broadening of the radiation spectrum and the appearance of a large number of combination components, or randomization of the radiation. It is observed that an increase in the length of the interaction region is accompanied by a considerable increase in the efficiency per electron. This increase is due to successive (relay) interactions between the electrons and the spectral components of the radiation with decreasing phase velocities.

* * *

INTRODUCTION

An ever-increasing number of papers using the concept of deterministic chaos [1-4] to treat erratic turbulent behavior in distributed systems has appeared in recent years. This concept can be most appropriately applied to self-excited oscillating systems, where (within finite limits) the steady-state regime is independent of the initial conditions. Such turbulence can be treated mathematically as a strange attractor in the corresponding infinite-dimensional phase space. In particular, similar stochastic self-excited oscillating modes occur in distributed microwave electron oscillators with various feedback mechanisms [2, 5-8]; examples of such oscillators include backward-wave-tubes (which make use of absolute instability in the electron beams) and resonant oscillators (resonator-based oscillators where the beam instability is convective in nature, and additional feedback must be introduced in order for self-excited oscillation to occur).

Also of interest (in addition to self-excited oscillating systems) are systems with pure convective instability. Included in systems of this class are hydrodynamic boundary layer flows and a wide variety of electron-beam systems with the group velocity of the resulting electromagnetic waves in the same direction as the forward velocity of the particles. When there is no external feedback, a nontrivial stationary (time-stationary) regime can only be achieved when some external perturbation is present. If this perturbation is concentrated at some point in the medium (or is localized within some small region), our discussion should then focus on the spatial evolution of the perturbations, which can lead not only to an increase in the intensity of the perturbations but also to an enrichment of the perturbation spectrum, which (given a sufficiently long interaction length) can lead to randomization of the spectrum. A similar transition to turbulence has been observed experimentally in boundary-layer flows [9] and in the relaxation of electron beams in a plasma [10].

In the present paper, we shall carry out a theoretical study of the randomization of an electromagnetic wave interacting with an electron beam in systems with convective instability. Unlike the large number of papers in which the transformation of a complex signal in electron-beam systems has been studied using a spectral approach (the fundamental frequency method [11-13]), we shall use a space-time approach, assuming that the total width of the signal spectrum

*Originally published in Radiotekhnika i elektronika, No. 4, 1989, pp. 821-829.

ISSN8756-6648/89/0016-0038$7.50/0

is small compared with the carrier frequency,* and we can use the equations in [5, 8] for the slowly varying envelope of the signal.

1. BASIC EQUATIONS

We shall write the field due to the electromagnetic wave in the form

$$E = \mathrm{Re}[A(z, t)e^{i\omega_0 t - ik_0 z}],$$

where $A(z, t)$ is the slowly varying complex amplitude, ω_0 is the carrier frequency, and $k_0 = k(\omega_0)$ is the wave number. For systems with inertial particle bunching [15] (plasma-beam systems and "O"-type microwave electron devices) the electron-wave interaction may be described using the following universal system of equations [7, 8] (assuming that the relative changes in particle energy are small):

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_g}\frac{\partial}{\partial t}\right)A - i\frac{\nu}{2}\frac{\partial^2 A}{\partial t^2} = \frac{\omega_0}{c}\frac{I}{\frac{mc^3}{e}}\frac{\kappa}{N}J(z, t),$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_{\parallel 0}}\frac{\partial}{\partial t}\right)w = \frac{\omega_0}{c}\kappa\mathrm{Re}(Ae^{i\theta}),$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{v_{\parallel 0}}\frac{\partial}{\partial t}\right)\theta = \frac{\omega_0}{c}(\Delta - \mu w) \qquad (1)$$

with boundary conditions

$$A|_{z=0} = A_0(t), \quad w|_{z=0} = 0, \quad \theta|_{z=0} = \theta_0 \in [0, 2\pi], \qquad (2)$$

where $v_g = d\omega_0/dk_0$ is the group velocity of the electromagnetic wave, N is the norm of the wave field, $\nu = d^2k_0/d\omega_0^2$ is a parameter describing the dispersive spreading of the wave, $v_{\parallel 0}$ is the unperturbed translational velocity of the electrons, $w = 1 - E/E_0$, E and E_0 are the initial and final electron energies, θ and θ_0 are the current and initial phases of the electrons in the wave field, $J = 1/\pi \int_0^{2\pi} e^{-i\theta} d\theta_0$ is the amplitude of the first harmonic in the high-frequency beam current, κ is the coupling coefficient between the electrons and the wave, and μ is a parameter describing the inertial bunching of the electrons (values of κ and μ for devices based on various stimulated emission mechanisms have been given by Bratman et al. [15]). Note that both the short-range Coulomb interaction between the electrons and the initial scatter in the particle velocities was neglected when writing (1).

Setting $v_{\parallel 0} \neq v_g$ in Eqs. (1) and (2), we see that it is convenient to change to the new independent variables

$$Z = \frac{\omega_0}{c} zC, \quad \tau = \frac{\omega_0}{c}\left(t - \frac{z}{v_{\parallel 0}}\right)C\left(\frac{1}{v_{\parallel 0}} - \frac{1}{v_g}\right)^{-1},$$

with the result that the equations are reduced to a form which contains the smallest possible number of independent parameters:

---

*The evolution of complex signals in broadband traveling-wave tubes was studied within the framework of a space-time approach in [14] without this assumption, which enables one to separate the signal into a carrier and an envelope.

$$\frac{\partial a}{\partial Z} + \frac{\partial a}{\partial \tau} - i\frac{\hat{\nu}}{2}\frac{\partial^2 a}{\partial \tau^2} = J,$$

$$\frac{\partial u}{\partial Z} = \mathrm{Re}(ae^{i\theta}), \quad \frac{\partial \theta}{\partial Z} = \delta - u,$$

$$a|_{Z=0} = a_0, \quad u|_{Z=0} = 0, \quad \theta|_{Z=0} = \theta_0 \in [0, 2\pi]. \tag{3}$$

where

$$a = \kappa\mu C^{-2}A, \quad u = \mu w C^{-1}, \quad \delta = \frac{\omega_0}{c}\Delta C^{-1},$$

$$\hat{\nu} = \omega_0 c\nu C\left(\frac{c}{v_{\parallel 0}} - \frac{c}{v_g}\right)^{-2}, \quad C = \left(eI/mc^3\frac{\kappa^2\mu}{N}\right)^{1/3}$$

is the generalized Pierce parameter. We shall characterize the efficiency with which the energy of the electron beam is transformed into energy of electromagnetic radiation using the reduced efficiency

$$\eta = \frac{1}{2\pi}\int_0^{2\pi} u\,d\theta_0.$$

We shall neglect the dispersive smearing of the wave and assume $\hat{\nu} \ll 1$. It should be noted, however, that the term responsible for dispersive smearing becomes most important when the dispersion curves for the beam and the electromagnetic waves are tangent to one another, with $v_{\parallel 0} \approx v_g$ and the second term on the left-hand side of Eq. (3) becoming negligible on the other hand. The situation where the dispersion curves are tangent to one another (which is characterized by a wide frequency bandwidth of interaction ($\Delta\omega/\omega_0 \sim \sqrt{C/\omega_0 c\nu}$), requires special discussion based upon a parabolic-type equation.

2. THE SATELLITE INSTABILITY IN THE SINGLE-FREQUENCY INTERACTION MODE

If a monochromatic perturbation $a_0 = \hat{a}_0 e^{i\Omega\tau}$ is applied at the boundary of the medium ($Z = 0$), the time dependence in (3) can be eliminated using the change of variables $a = \hat{a}(Z)e^{i\Omega(\tau - Z)}$, $\theta = \hat{\theta}(Z) - \Omega(\tau - Z)$, leading to the well-known problem of the amplification of a monochromatic wave [16]:

$$\frac{d\hat{a}}{dZ} = J, \quad \frac{d^2\theta}{dZ^2} = \mathrm{Re}(\hat{a}e^{i\theta}),$$

$$\hat{a}|_{Z=0} = \hat{a}_0, \quad \hat{\theta}|_{Z=0} = \hat{\theta}_0, \quad \left.\frac{d\theta}{dZ}\right|_{Z=0} = \delta - \Omega. \tag{4}$$

A numerical treatment of this problem can be found in numerous sources (see, for example, [13, 16]). One characteristic feature of the solution is the development of a beam instability within the electron bunches during the initial (linear) stage. The bunches are then trapped by the wave as they execute periodic oscillations from the retarding phase of the wave to the advancing phase and back again; this means that the energy exchange between the wave and bunches is periodic in Z. Only when the interaction length is very long do the bunches become blurred and the variations in wave amplitude become less severe (due to the fact that the oscillation frequency of the trapped particles is a function of the energy). We can thus use an approximation where the electrons are all assumed to be grouped into a single "macroparticle"* to analyze

---

*The satellite instability has also been discussed within the framework of other models (see [8, 13]).

the stability of the stationary interaction mode. The interaction in the system consisting of the wave and macroparticle may be described using the following equations:

$$\frac{\partial a}{\partial Z} + \frac{\partial a}{\partial \tau} = 2e^{-i\theta},$$
$$\frac{\partial^2 \theta}{\partial Z^2} = \mathrm{Re}(ae^{i\theta}). \tag{5}$$

It will then be convenient to make the change of variables $B = ae^{i\theta}$ and cast Eqs. (5) in the form

$$\frac{\partial B}{\partial Z} + \frac{\partial B}{\partial \tau} = 2 + iB\left(\frac{\partial \theta}{\partial Z} + \frac{\partial \theta}{\partial \tau}\right), \quad \frac{\partial^2 \theta}{\partial Z^2} = \mathrm{Re}\, B. \tag{6}$$

From (6), we can then obtain the following result in the stationary case:

$$\frac{\partial B}{\partial \tau} = \frac{\partial \theta}{\partial \tau} = 0,$$

using the integral

$$|B|^2 - \frac{d\theta}{dZ} = G,$$

(which is the law of conservation of energy):

$$\frac{dB_0}{dZ} = 2 + iB_0(|B_0|^2 - G). \tag{7}$$

Equation (7) has the equilibrium state $B_0 = iD$ given by the equation

$$2 - D(D^2 - G) = 0, \tag{8}$$

as well as solutions which are periodic in the longitudinal coordinate.

We shall now discuss the stability of the stationary solution $B_0(Z)$, $\theta_0(Z)$ against small, generally speaking nonstationary perturbations, i.e., perturbations whose frequencies differ from that of the original wave:

$$B = B_0(Z) + b(Z, \tau) + id(Z, \tau)$$
$$\theta = \theta_0(Z) + \vartheta(Z, \tau). \tag{9}$$

Linearizing (8), we obtain the following equations for the perturbations:

$$\left(\frac{\partial}{\partial Z} + \frac{\partial}{\partial \tau}\right) b = -\mathrm{Im}\, B_0\left(\frac{\partial \vartheta}{\partial Z} + \frac{\partial \vartheta}{\partial \tau}\right) - d\,\frac{\partial \theta_0}{\partial Z},$$
$$\left(\frac{\partial}{\partial Z} + \frac{\partial}{\partial \tau}\right) d = \mathrm{Re}\, B_0\left(\frac{\partial \vartheta}{\partial Z} + \frac{\partial \vartheta}{\partial \tau}\right) + b\,\frac{\partial \theta_0}{\partial Z}, \quad \frac{\partial^2 \vartheta}{\partial Z^2} = b. \tag{10}$$

Since the coefficients of linear system (10) are independent of the time τ, we may seek a solution of the form $b = \bar{b}(Z)e^{-i\Omega\tau}$, etc. In the special case where $B_0 = iD$, we have $d\theta_0/dZ = 0$, i.e., equilibrium state (8) corresponds to unperturbed motion. We can also assume $\bar{b}, \bar{d}, \bar{\vartheta} \propto e^{i\kappa Z}$, leading to the following dispersion equation for the perturbation wave number:

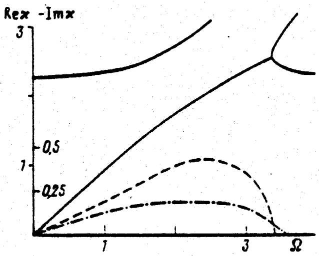

Fig. 1. Real part (the solid line) and imaginary part (the dash-dot line) of κ obtained by solving dispersion equation (11) for $D = 5$. The dash-dot line is the spatial increment of the instability in the spatially periodic single-frequency mode ($\delta = 1$, $\hat{a}_0 = i$).

$$\kappa^2(\kappa-\Omega)^2 - D(\kappa-\Omega)^2 - (4/D^2)\kappa^2 = 0, \tag{11}$$

which includes the unique parameter D. It is interesting to note that Eq. (11) is identical to the dispersion equation for low-amplitude waves in a system consisting of a plasma and a beam [17]. On the other hand, dispersion equation (11) can also be rearranged in the following form:

$$[(\kappa-\Omega)^2 - 4/D^2](\kappa^2 - D) = 4/D. \tag{11a}$$

This representation enables us to treat the instability under consideration as being due to coupling between the electromagnetic satellite waves ($\kappa - \Omega = \pm 2/D$) and the waves in the beam ($\kappa = \pm\sqrt{D}$). The waves in the beam are due to the excitation of particle oscillations in the potential well formed by the main wave (these oscillations have frequency $\sqrt{D}$) under the influence of the satellite waves. Unfortunately, Eq. (11) does not have a small parameter which would enable us to use the coupled-wave method to obtain an analytic expression for the increment.

Equation (11) was analyzed numerically, with the analysis indicating that there is always a range of spatially unstable ($\mathrm{Im}\,\kappa < 0$) satellite-wave frequencies $0 < \Omega < \Omega_{max}$. Note that there is an optimum difference between the satellite-wave frequency and the frequency of the initial wave, Ω_{opt}, for which the increment is maximum.

In the more general case where $B_0(Z)$ and $\theta_0(Z)$ are periodic functions of the longitudinal coordinate, we may seek a solution of (10) of the form $\bar{b}(Z) \sim \hat{b}(Z)e^{i\kappa Z}$ (with similar expressions holding for $\bar{a}$ and $\bar{\vartheta}$), where $\hat{b}(Z)$ is a periodic function of Z. The increment κ may be determined numerically by simultaneous solution of Eqs. (4) and (10). The results of the numerical analysis shown in Fig. 1 indicate that the mode with spatially periodic stationary energy exchange between the macroparticle and wave is also unstable with respect to the excitation of satellite waves.

3. MULTIFREQUENCY MODE

The nature of the nonlinear stage in the satellite instability described above is determined by the form of the perturbations applied at the boundary $Z = 0$. If a perturbation with periodic time modulation, i.e., $a_0(0, \tau) = a_0(0, \tau + T)$, is applied at the boundary, the solution will be a periodic function of time at all Z due to the fact that Eq. (3) is invariant with respect to a shift in time. In frequency terms, this means that no spectral components other than harmonics of the frequency $\Omega = 2\pi/T$ can appear: $a(Z,\tau) = \sum_n a_n(Z)e^{in\Omega\tau}$. Thus, the periodic waves do not become stochastic. If, on the other hand, a noise perturbation with a continuous spectrum is applied at the boundary, it will begin to be amplified and broadened by the secondary instability discussed above, and then enters a stationary turbulent regime for

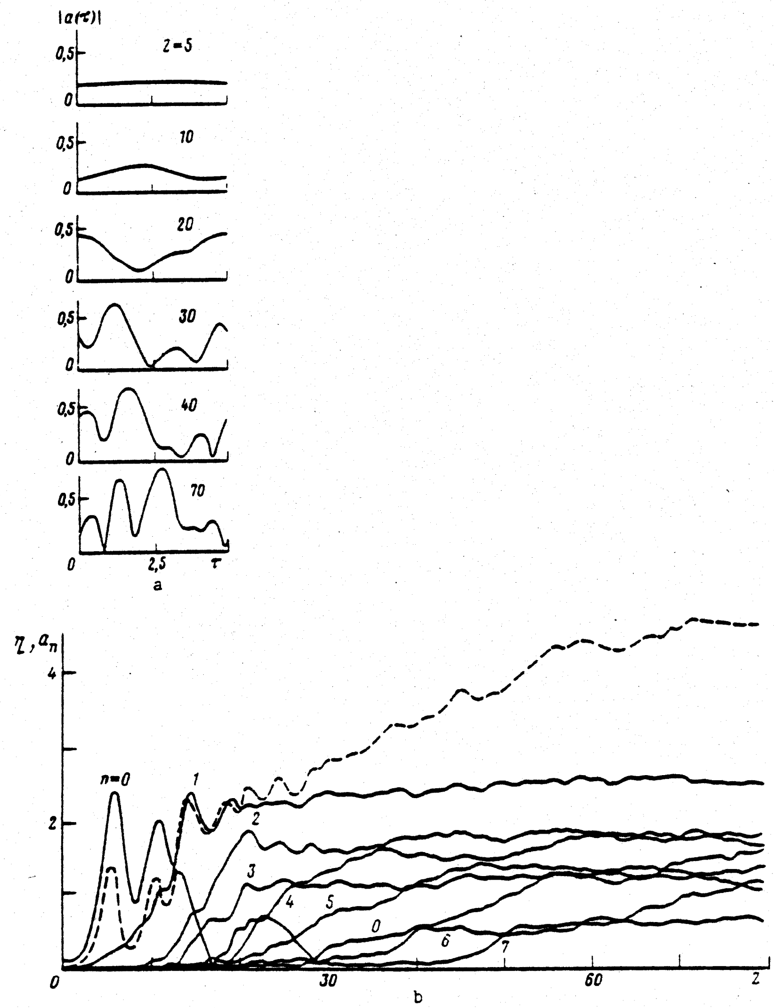

Fig. 2. Evolution of the envelope (a) and spectrum (b) of a periodic signal along the longitudinal coordinate. The dashed line indicates the reduced efficiency; T = 4.8; Δ = 0.

large Z. A borderline situation occurs when a quasi-periodic perturbation containing the incommensurable frequencies Ω_1 and Ω_2 is applied to the input. Their interaction then leads to the growth of combination frequencies of the form $n\Omega_1 + m\Omega_2$ during the nonlinear stage and to the formation of a nearly continuous spectrum (for sufficiently long lengths). This type of randomization of an input signal was studied in [18] using a chain of nonlinear amplifiers as an example.

We shall now describe the results of our numerical modeling of the multifrequency processes. Figure 2 shows the evolution of a periodic input signal whose spectrum initially contains the harmonics $n = 0$ ($a_0 = 0.1$) and $n = \pm 1$ ($a_{\pm 1} = 0.01$). The process unfolds as follows. During the initial phase, the harmonic $n = 0$ (i.e., the one with the largest linear increment) experiences exponential growth, accompanied by bunching of the electron beam and the formation of electron bunches. The bunches are then trapped by the wave, and this gives rise to a periodic exchange of energy between the bunches and the wave. Secondary instabilities and higher-order harmonics begin to develop during this stage. These effects then lead to disruption of the electron bunches and a substantial slowing of the energy exchange between the particles and wave.

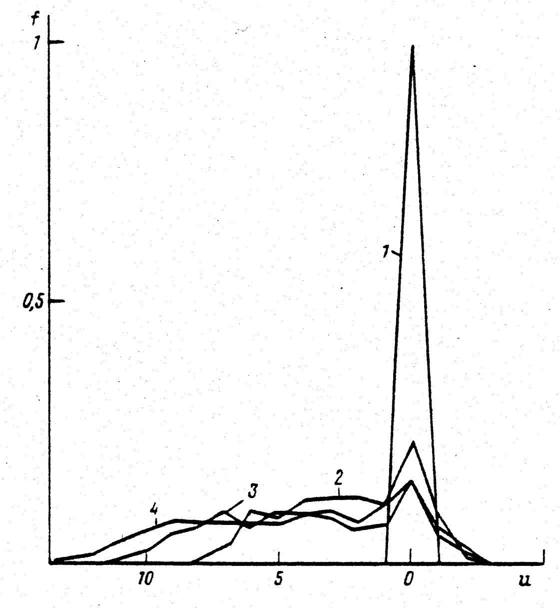

Fig. 3. Evolution of the distribution function for the electron energy (T = 4.8 and Δ = 0) for Z = 0 (*1*), 25 (*2*), 50 (*3*), 75 (*4*).

During the next (and final) stage, the particles even enter a weakly turbulent state. A plateau (whose edge gradually shifts to lower electron energies—see Fig. 3) is formed in the electron energy (velocity) distribution. The fact that the derivative of the velocity distribution function is nonnegative at the edge of the plateau leads to amplification of the additional slow harmonics with phase velocities that turn out to be similar to the velocities of the particles at the edge of the plateau. These harmonics have virtually random phases. This stage in the process may be described using quasi-linear diffusion equations [19] as long as we assume that the initial "seed" for each of the harmonics is specified at the boundary. The numerical simulation we discuss here did not include the seeds for the higher-order harmonics, which originated in combination interactions between the lower-order harmonics. Note that the efficiency per electron increases as the plateau on the distribution function increases in size; this means that the efficiency may be much higher (five times higher in Fig. 2a) for amplification of a complex signal than for a monochromatic signal. The mechanism for this increase in efficiency is quite obvious: once an electron has interacted with one harmonic and transferred some of its energy to that harmonic, it becomes trapped by the next harmonic (which has a lower phase velocity), and so on.

Thus, the secondary instabilities and the increasing complexity of the spectrum of fields which act on the electrons lead to rapid disruption of the particle bunches and randomization of the electron motions. Moreover, even though the amplitudes and phases of the individual harmonics in the wave field show a nearly random variation along the Z axis, and the radiation spectrum has a total width much greater than the linear amplification bandwidth ($\Delta\omega/\omega_0 \gg C$), the waves do not become fully randomized in the case under discussion (i.e., in the case of periodic boundary conditions), since the time spectrum of the signal remains discrete.

As was noted above, it turns out that waves may become randomized when the field at the boundary is quasi-periodic; in this case, the process of randomization involves the onset and growth of combination harmonics which densely fill a certain spectral region. However, computer modeling of the amplification of two (or more) incommensurable harmonics requires the integration of Eqs. (3) over an infinite (or at least very long) time interval. However, the process by which the spectrum is filled can also be illustrated within the framework of the time-periodic problem if the higher-order harmonics (rather than the first harmonic) are specified in addition to the fundamental harmonic. In particular, calculations carried out for the case where $a_0 = a_3 = a_5 = 0.1$ indicated that the onset of the satellite instability is immediately followed by growth of the combination frequencies and filling of the spectrum with the n = 1, 2, 4, 6... harmonics omitted in the original signal.

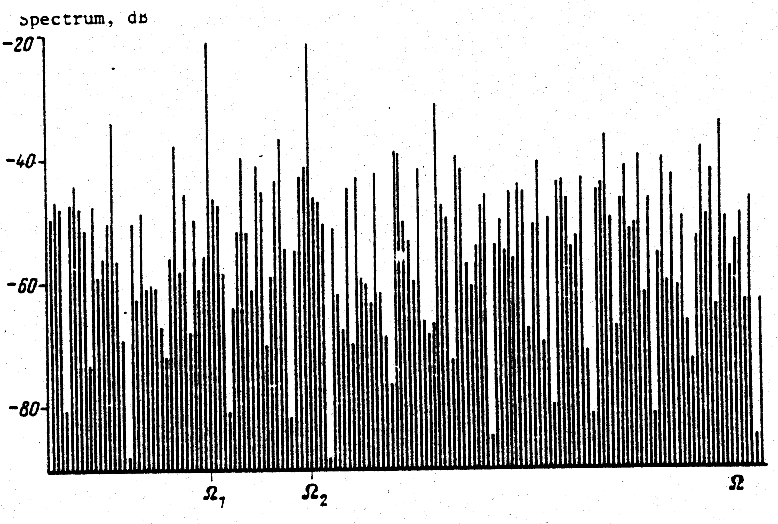

Fig. 4. Filling of the signal spectrum by combination harmonics in the klystron model.

The process by which a quasi-periodic input signal becomes randomized may be described more clearly within the framework of the klystron model [7, 8], where we assume that the interaction between the wave and the electrons occurs in two narrow intervals separated by a region where the electrons drift freely. Substituting the coupling coefficient between the electrons and wave (in the form $\kappa = \kappa_1(z) + \kappa_2\delta(z - l)$, where δ is the delta function) into Eqs. (1), and integrating the resulting equations, we obtain the following relationship between the input and output signal amplitudes:

$$a(\xi, L) = a(\xi, 0) + ie^{-i\psi} \left\{\frac{a}{|a|} J_1(X|a|)\right\} \Bigg|_{\substack{Z=0 \\ \xi'=\xi-L}}, \tag{12}$$

where J_1 is the first-order Bessel function

$$a = \frac{\omega_0}{c}\,\frac{I}{mc^3/e}\,\frac{\kappa_2}{N} A, \quad X = \frac{\mu\omega_0^2}{c^2}\,\frac{\kappa_1\kappa_2 LI}{Nmc^3/e}, \quad L = \frac{\omega_0}{c} l,$$

$$\xi = \omega_0\left(t - \frac{z}{v_g}\right)\left(\frac{c}{v_g} - \frac{c}{v_{\parallel 0}}\right)^{-1}, \quad \psi = \Delta l.$$

If the input signal contains incommensurable harmonics,

$$a(\xi, 0) = a_0 + a_1\cos\Omega_1\xi + a_2\cos\Omega_2\xi,$$

and we then obtain (provided X is large) a signal containing all possible combination frequencies $n\Omega_1 + m\Omega_2$ (Fig. 4), i.e., a practically random signal.

REFERENCES

1. Gaponov-Grekhov, A. V. and M. I. Rabinovich. Nelineynye volny. Bifurkatsii i struktury (Nonlinear Waves: Bifurcations and Structure). Nauka Press, Moscow, 1987.
2. Kislov, V. Ya., N. N. Zalogin, and E. A. Myasin. Radiotekhnika i elektronika, 24, No. 6, p. 1118, 1979.

3. Pikovskiy, A. S. Pis'ma v ZhETP, 40, No. 6, p. 217, 1984.
4. Farmer, J. D. Physica D., 4, No. 3, p. 366, 1982.
5. Ginzburg, N. S., T. N. Fedoseeva, and S. P. Kuznetsov. Izv. vuz. Radiofizika, 21, No. 7, p. 1037, 1978.
6. Bezruchko, B. P., S. P. Kuznetsov, and D. I. Trubetskov. Pis'ma v ZhETP, 29, No. 3, p. 180, 1979.
7. Ginzburg, N. S. and S. P. Kuznetsov. Relyativisticheskaya vysokochastotnaya elektronika (Relativistic High-Frequency Electronics). Institut Prikladnoy Fiziki, Gorkiy, No. 2, p. 101, 1981.
8. Ginzburg, N. S. and M. I. Petelin. International Journal of Electronics, 59, No. 3, p. 291, 1986.
9. Swinney, H. L. and J. P. Gollub. Hydrodynamic Instabilities and the Transition to Turbulence. Springer Verlag, New York, 1985.
10. Artsimovich, L. A. and R. Z. Sagdeev. Fizika plazmy dlya fizikov (Plasma Physics for Physicists). Atomizdat Press, Moscow, 1979.
11. Zhelezkovskiy, B. E. and E. V. Kal'yanov. Mnogochastotnye rezhimy v priborakh SVCh (Multifrequency Modes in Microwave Devices). Svyaz Press, Moscow, 1978.
12. Kats, A. M., V. P. Kudryashev, and D. I. Trubetskov. Signal v lampakh s begushchey volnoy (Signal in Traveling-Wave Tubes). Saratovsk. Univ. Press, Saratov, 1984.
13. Shapiro, V. D. and V. I. Shevchenko. Izv. vuz. Radiofizika, 19, Nos. 5-6, p. 767, 1976.
14. Man'kin, i. A. and V. G. Shkol'nikov. Radioelektronika, 26, No. 9, p. 1918, 1981.
15. Bratman, V. L., N. S. Ginzburg, N. F. Kovalev, et al. Relyativisticheskaya vysokochastotnaya elektronika (Relativistic High-Frequency Electronics). Institut Prikladnoy Fiziki, Gorkiy, No. 1, p. 249, 1979.
16. Vaynshteyn, L. A. and V. A. Solnstsev. Lektsii po elektronike SVCh (Lectures on Microwave Electronics). Sovetskoye Radio Press, Moscow, 1973.
17. Mikhaylovskiy, A. B. Teoriya plazmennykh neustoychivostey (Theory of Plasma Instabilities). Vol. 1. Atomizdat Press, Moscow, 1975.
18. Pikovskiy, A. S. Izv. vuz. Radiofizika, 30, No. 9, p. 756, 1987.
19. Vedenov, A. A. E. P. Velikhov, and R. Z. Sagdeev. Yadernyy Sintez. Prilozhenie (Nuclear Synthesis. Applications), Part 2. 1962.

□◆◆◆□