RENORMALIZATION GROUP FOR THE RESPONSE FUNCTION AND SPECTRUM OF THE PERIOD-DOUBLING SYSTEM

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The renormalization transformation for the response function and the spectrum of one-dimensional mapping, exhibiting a period-doubling transition to chaos, are presented. The transformations include frequency doubling leading to chaotic behaviour instead of a usual fixed point. Scaling of the response function and the spectrum is of a statistical nature and only some average characteristics can be described by universal constants.

1. Since Feigenbaum's discovery of the quantitative universality for the transition to chaos through period-doubling bifurcations there has been great enthusiasm in application of renormalization group (RG) methods to dynamical systems. The Feigenbaum systems, i.e. one-dimensional mappings exhibiting period-doubling bifurcations, were investigated most thoroughly [1–3]. In particular, universal constants for external noise scaling [4–6], two-system interaction [7] and period doubling in continuous media [8,9] were obtained. In this paper we develop a RG approach to scaling of the response function and the spectrum for period-doubling bifurcations. The peculiarity of this situation is that a RG transformation has a strange attractor instead of a fixed point.

The power spectrum has a large physical significance and can easily be obtained experimentally. Several universal constants for scaling of the spectrum were obtained numerically rather than by a RG method in refs. [10–12]. Approximate scaling laws for the spectrum form were also formulated in refs. [13–18] and approximate values of the constants were derived. Our RG approach permits us to obtain some constants as eigenvalues of the renormalization transformation.

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The response function is not so easy to obtain experimentally as the spectrum. However, it was theoretically investigated in refs. [19–23]. The influence of an external periodic force on period doublings was considered numerically in ref. [24] and our RG approach is consistent with those results (qualitative experiments on period doubling in the presence of a periodic external force were performed in ref. [25]). It should be noted that an attempt to develop a RG method for a periodically forced system was made in refs. [22,23]. The significance of frequency doubling in this RG was mentioned recently independently from our paper in refs. [26,27].

2. Periodic-doubling bifurcations are described by a one-dimensional mapping. Let us denote the mapping for a critical parameter value by $f(x)$. Suppose that the mapping is performed by a small periodic force with a field-dependent amplitude Q:

$$y_{n+1}=f(y_n)+\epsilon Q(y_n)\exp(2\mathrm{i}\pi\omega n)\,,\quad \epsilon\ll 1\,.$$

After linearization near an unperturbed trajectory x_n we obtain for a small disturbance r_n:

$$r_{n+1}=f(x_n)r_n+Q(x_n)\exp(2\mathrm{i}\pi\omega n)\,.$$

The spectrum of some function of x looks like

$$S(\omega)=\frac{1}{N}\sum_{n=1}^{N}H(x_n)\exp(2\mathrm{i}\pi\omega n)\,.$$

Thus, to describe the response function and spectrum, it is convenient to consider simultaneous iterations of x and two additional variables,

$$x_{n+1}=f(x_n)\,,$$
$$r_{n+1}=f'(x_n)r_n+Q(x_n)\exp(2i\pi\omega n)\,,$$
$$s_{n+1}=s_n+H(x_n)\exp(2i\pi\omega n)\,. \tag{1}$$

Variables r and s correspond to the non-normalized response function and the spectrum, respectively. The functions Q and H define the external force amplitude and the spectral variable. The normalized response function $R_m(\omega)$ and the spectrum $S_m(\omega)$ for a cycle of period 2^m are related to r and s as

$$R_m(\omega)=(-\alpha)^m r_{2^m}\,,\quad S_m(\omega)=2^m s_{2^m}\,, \tag{2}$$

where $\alpha=2.5029...$ is the Feigenbaum constant for the scaling of x. Applying (1) twice and scaling the variables $x=-\bar{x}/\alpha$, $r=-\bar{r}/\alpha$, $s=2\bar{s}$ we obtain mappings of the same type as (1) but with renormalized functions f, Q and H and the renormalized frequency ω:

$$\bar{f}(x)=\hat{D}f(x)=-\alpha f^2(-x/\alpha)\,, \tag{3a}$$
$$\bar{Q}(x)=-\alpha Q(-x/\alpha)f'(f(-x/\alpha))$$
$$-\alpha Q(f(-x/\alpha))\exp(2i\pi\omega)\,, \tag{3b}$$
$$\bar{H}(x)=\tfrac{1}{2}H(-x/\alpha)$$
$$+\tfrac{1}{2}H(f(-x/\alpha))\exp(2i\pi\omega)\,, \tag{3c}$$
$$\bar{\omega}=2\omega\quad(\mathrm{mod}\ 1)\,. \tag{3d}$$

The transformation (3a) has a stable fixed point, which is the Feigenbaum function g [1,2],

$$\lim_{l\to\infty}\hat{D}^l f(x)=g(x)\,.$$

Thus we may substitute g for f in transformations (3b), (3c). As a result, introducing the "renormalization time" l, we obtain

$$Q_{l+1}=\hat{M}(\omega)Q_l(x)\equiv-\alpha Q_l(-x/\alpha)g'(g(-x/\alpha))$$
$$-\alpha Q_l(g(-x/\alpha))\exp(2i\pi\omega_l), \tag{4}$$
$$H_{l+1}=\hat{L}(\omega)H_l\equiv\tfrac{1}{2}H_l(-x/\alpha)$$
$$+\tfrac{1}{2}H_l(g(-x/\alpha))\exp(2i\pi\omega_l)\,, \tag{5}$$
$$\omega_{l+1}=2\omega_l\quad(\mathrm{mod}\ 1)\,. \tag{6}$$

3. Let us first consider the renormalization transformation for the response function (4), (6). It should be noted that eq. (4) was obtained in ref. [22] but without eq. (6); the system (4), (6) was derived independently recently in ref. [27]. In order to investigate the iterations of (4), (6) it is convenient to introduce a normalized amplitude $\tilde{Q}_l(x)$ (using the condition $\tilde{Q}_l(0)=1$) and define the amplitude factor q_l:

$$\tilde{Q}_{l+1}(x)=\hat{M}(\omega_l)\tilde{Q}_l(x)/q_l\,,$$
$$q_l=[\hat{L}(\omega_l)\tilde{Q}_l]\,|_{x=0}\,. \tag{7}$$

The amplitude factor q_l shows how the amplitude Q_l scales per unit of "renormalization time". The transformation of the normalized function $\tilde{Q}_l$ is nonlinear. The main point is that the transformation dynamics is chaotic. It is easily seen that chaos arises from the doubling transformation for the frequency (6). Correspondingly, the strange attractor in the infinite-dimensional phase space is the well-known Smale–Williams attractor [28]. Its projections on the planes $(|q_l|, \omega_l)$ and $(\arg q_l, \omega_l)$ look like a product of a Cantor set and an interval (see fig. 1a). (Numerically, iterations of transformation (4)–(6) were made using the known polynomial representation for g [1,29].)

The scaling properties of the response function $R_m(\omega)$ are obtained in the following way. Suppose that the external force with the normalized amplitude $\tilde{Q}_0(x)$ is applied to a 2^m-cycle. Then employing the renormalization we have

$$R_m(\omega,\tilde{Q}_0)=q_0(\omega)R_{m-1}(2\omega,\tilde{Q}_1)=...$$
$$=q_0(\omega)q_1(2\omega)...q_{m-1}(2^{m-1}\omega)R_0(2^m\omega,\tilde{Q}_m)\,. \tag{8}$$

Note that the points q_0, q_1, ... do not lie on the attractor but approach it as $m\to\infty$. Thus for large m the quantity $q_m(2^m\omega)$ lies on the attractor. It follows from (8) that

$$\frac{R_{m+1}(\omega,\tilde{Q}_0(x))}{R_m(\omega,\tilde{Q}_0(x))}$$
$$=q_m(2^m\omega)\,\frac{R_0(2^{m+1}\omega,\tilde{Q}_{m+1}(x))}{R_0(2^m\omega,\tilde{Q}_m(x))}\,. \tag{9}$$

The ratio in the r.h.s. of eq. (9) depends on the normalization condition (7). Hence the ratio $R_{m+1}(\omega,$

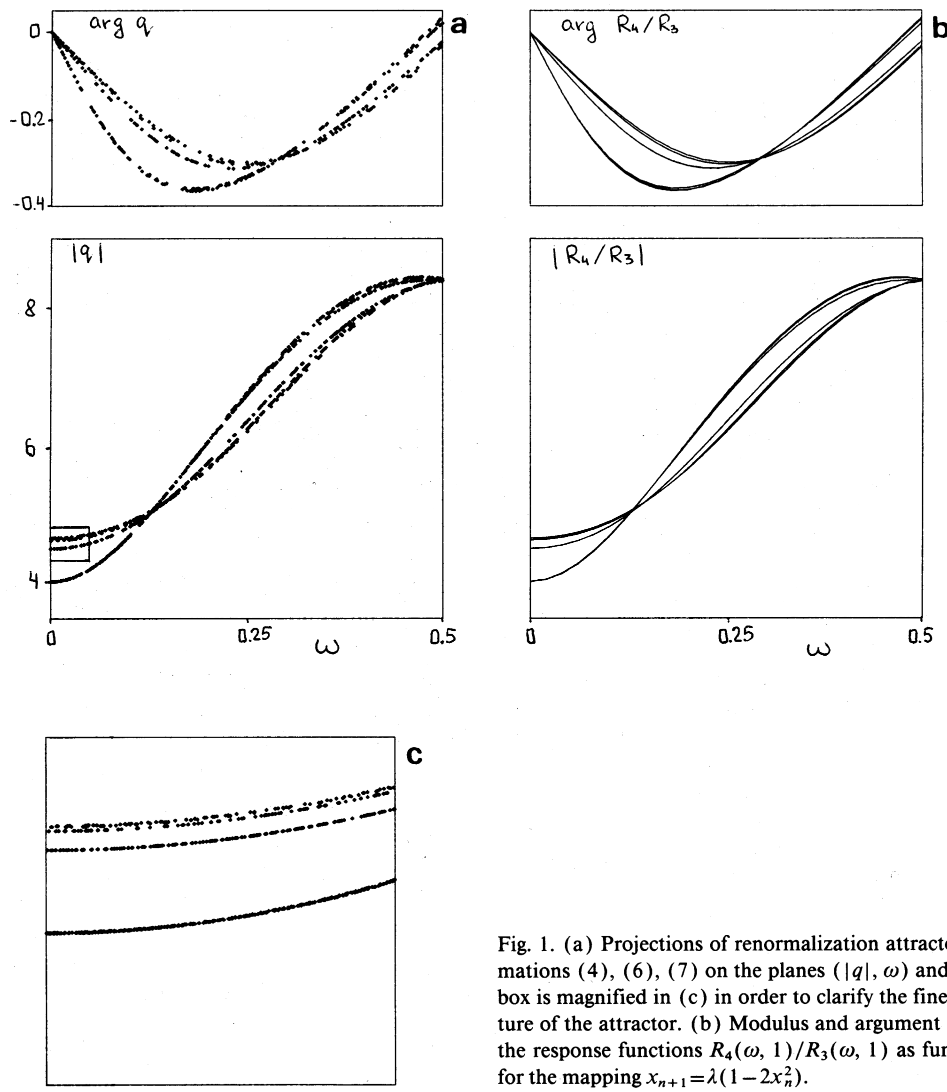

Fig. 1. (a) Projections of renormalization attractor in transformations (4), (6), (7) on the planes ($|q|$, ω) and ($\arg q$, ω). A box is magnified in (c) in order to clarify the fine Cantor structure of the attractor. (b) Modulus and argument of the ratio of the response functions $R_4(\omega, 1)/R_3(\omega, 1)$ as functions of $2^3\omega$ for the mapping $x_{n+1}=\lambda(1-2x_n^2)$.

$1)/R_m(\omega, 1)$ (see fig. 1b) should not be compared directly with $q(2^m\omega)$. One see that plots of $R_{m+1}(\omega, 1)/R_m(\omega, 1)$ and $q(2^m\omega)$ are very similar. It should be noted that in refs. [22,23] a single-valued function $\kappa(\omega)$ was obtained as an eigenfunction of (4); this function obviously cannot be compared with the many-valued ratio of the response functions (see fig. 1b).

Let us consider special cases when the "renormalization trajectory" belongs to exceptional subsets of the renormalization attractor, i.e. fixed or periodic points.

$\omega=0$ is a fixed point of transformation (6) at which eq. (4) becomes the well-known Feigenbaum equation [1] with eigenvalue $q_m=\delta=4.669...$. This means that the effect of the constant external force is equivalent to a shift of the control parameter. The same is true for the force with $\omega=2^{-p}k$: this external force has a nontrivial effect only at the first few bifurcations.

If $\omega=p_1/p_2$ is rational, then the sequences ω_l and q_l are periodic with some period p. The eigenvalue of the renormalization transformation at p iterations is

Table 1

$\omega=0, m=1$		$\omega=\frac{1}{3}, m=2$		$\omega=\frac{1}{7}, m=3$			$\omega=\frac{2}{5}, m=4$		$\omega=\frac{4}{9}, m=6$	
n	R_n/R_{n-m}	n	R_n/R_{n-m}	n	$\|R_n/R_{n-m}\|$	$\arg R_n/R_{n-m}$	n	R_n/R_{n-m}	n	R_n/R_{n-m}
7	4.66915417	7	58.9599304	7	296.26953	−0.54923	7	2305.6494	8	56199.22
8	4.66919136	8	58.9605713	8	296.27783	−0.54925	8	2306.0625	9	56280.01
9	4.66919899	9	58.9605408	9	296.27832	−0.54925	9	2306.0395	10	56321.68
10	4.6692009	10	58.960556	10	296.27832	−0.54925	10	2306.05884	11	56319.79
B	4.6692015		58.9605665		296.27854	−0.54925		2306.0605		56319.11

$$B(\omega)=\prod_{l=0}^{p-1} q_l(2^l\omega) \tag{10}$$

and this value determines scaling of the response function (in (9) we have $\tilde{Q}_{m+p}(x)=\tilde{Q}_m(x)$, $2^{m+p}\omega=2^m\omega$ (mod 1)):

$$R_{m+p}(\omega)/R_m(\omega)=B(\omega)\,. \tag{11}$$

The validity of eq. (11) was checked numerically (see table 1). Eq. (11) means that the dynamics of the forced system does not vary if the time rescales by a factor 2^p and the external force scales by a factor $B(\omega)$.

The typical case when ω belongs to the set of unit measure in the interval $0<\omega<1$, i.e. ω is irrational, is most interesting. Here the values q_l are not repeated, so there is no scaling in the common sense. We may speak however of statistical scaling using the mean (averaged over the renormalization attractor) factor β:

$$\beta=\lim_{N\to\infty}\frac{1}{N}\sum_{l=0}^{N-1}\ln|q_l|\simeq 1.82\,. \tag{12}$$

The universal constant β was first obtained in ref. [21]. What does the statistical scaling mean? To answer this question let us consider the results of numerical experiments [24] on the periodically forced Feigenbaum system with fixed irrational frequency $\omega=(\sqrt{5}-1)/2$. On the place of parameters $(\lambda_c-\lambda, \epsilon)$ there are regions of regular 2^m-tori and of chaotic behaviour. In accordance with the statistical nature of scaling a border of chaos must run along a line $\ln\epsilon=-\kappa\ln(\lambda_c-\lambda)+\text{const}$, where $\kappa=\beta/\ln\delta=1.19$. In fig. 2 this line is broken. One can see that the border runs along this line (statistical scaling) but does not repeat itself (absence of usual scaling). An analogous scaling relation holds for a Lyapunov exponent Λ. The dependence of Λ on ϵ for a critical parameter value has the form $\Lambda\sim\epsilon^\chi$, where $\chi=\ln 2/\beta\simeq 0.38$. This prediction agrees with the numerical data presented in ref. [22]. It should be noted that the constant χ differs from the scaling constant for external random noise, used in refs. [22,23,26] for periodic forcing.

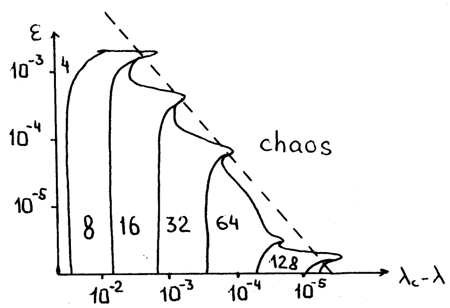

Fig. 2. Phase diagram of the quadratic map with parameter λ governed by a periodic external force with amplitude ϵ (reproduced from ref. [24]). The broken line has a slope κ.

At the end of this section we would like to point out that there exists an approximate scaling relation for the response function,

$$R_{n+1}(\omega)/R_n(\omega)\simeq\alpha^2-\alpha\exp(2i\pi\omega\times 2^n)\,. \tag{13}$$

This equation can easily be obtained from eq. (4) if one uses the approximation $Q\simeq$ const and takes into account that $g'(g(-x/\alpha))\simeq-\alpha$ for $x\simeq 0$.

4. Consider now the renormalization transformation for the spectrum (5), (6). The main difficulty in interpreting this transformation is the normalization condition for the function $H_l(x)$. Any constant added to $H_l(x)$ will not practically affect the varying part of $H_l(x)$. We propose to overcome this difficulty in the following way: only the varying part of $H_l(x)$ is important and this part corresponds to scaling of the new-born spectrum (i.e. spectral components appearing at the lth bifurcation), while the constant part of $H_l(x)$ corresponds to the old part of the spectrum. Thus it is convenient to define the amplitude factor h_l as

$$\tilde{H}_{l+1}=\frac{\hat{L}(\omega)\tilde{H}_l}{h_l}\,,\quad h_l=\left.\frac{\mathrm{d}^2[\hat{L}(\omega_l)\tilde{H}_l]}{\mathrm{d}x^2}\right|_{x=0}. \tag{14}$$

Here $\tilde{H}_l$ is the normalized amplitude function of x^2. The dynamics of the renormalization transformation for H_l is completely analogous to that of the response function. In phase space there is a renormalization attractor of the Smale–Williams type (see fig. 3a). As has been shown in refs. [13–15] the spectral components which emerge after the nth bifurcation are related as

$$\tilde{S}_n(\omega)=\frac{1}{2^{n-1}}\sum_{j=0}^{2^{n-1}-1}[x(j)-x(j+2^{n-1})]\times\exp(2\mathrm{i}\pi\omega j)\,. \tag{15}$$

Plots of $\tilde{S}_{n+1}(\omega)/\tilde{S}_n(\omega)$ as functions of $2^n\omega$ are shown in fig. 3b. A Cantor set arising at $n\to\infty$ does not coincide with that of fig. 3a due to its dependence on the normalization condition. However, for rational values of ω we may introduce the scaling function by analogy with (10),

$$C(\omega)=h(\omega)h(2\omega)\ldots h(2^{p-1}\omega) \tag{16}$$

and compare this function with $\tilde{S}_{n+p}(\omega)/\tilde{S}_n(\omega)$ (see table 2).

The fixed point $\omega=0$ is of special interest. The eigenvalue of the ω-independent renormalization transformation

$$H^0_{l+1}(x)=\tfrac{1}{2}[H^0_l(-x/\alpha)+H^0_l(g(-x/\alpha))] \tag{17}$$

equals $h^0=0.109033\ldots$. This constant describes the scaling of spectral line broadening beyond the transition point in systems with phase modulation

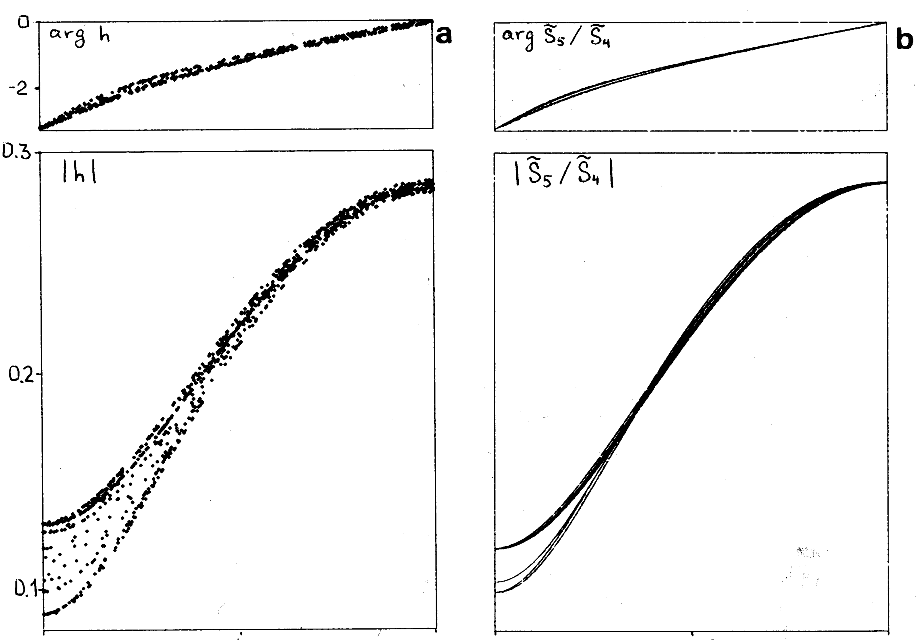

Fig. 3. (a) Projections of the renormalization attractor in transformations (5), (6), (16) on the planes ($|h|$, ω) and (arg h, ω). (b) Modulus and argument of the spectral function ratio $\tilde{S}_5(\omega)/\tilde{S}_4(\omega)$ as function of $2^4\omega$.

Table 2

$\omega=0, m=1$		$\omega=\frac{1}{3}, m=2$		$\omega=\frac{1}{7}, m=3$			$\omega=\frac{1}{5}, m=4$		$\omega=\frac{1}{9}, m=6$	
n	$\tilde{S}_n/\tilde{S}_{n-m}$	n	$\tilde{S}_n/\tilde{S}_{n-m}$	n	$\lvert\tilde{S}_n/\tilde{S}_{n-m}\rvert$	$\arg\tilde{S}_n/\tilde{S}_{n-m}$	n	$\tilde{S}_n/\tilde{S}_{n-m}$	n	$\tilde{S}_n/\tilde{S}_{n-m}$
12	0.10977286	7	0.06378961	7	0.01085161	−2.46999	7	0.2879982×10^{-2}	10	0.742399×10^{-4}
13	0.10847729	8	0.06379449	8	0.01085730	−2.46999	8	0.2883891×10^{-2}	11	0.742451×10^{-4}
14	0.10942394	9	0.06379020	9	0.01084847	−2.46936	9	0.2883844×10^{-2}	12	0.742837×10^{-4}
15	0.10872984	10	0.06379068	10	0.01084807	−2.46954	10	0.2885363×10^{-2}	13	0.742692×10^{-4}
C	0.10903311		0.06379026		0.01084848	−2.46955		0.2885226×10^{-2}		0.742696×10^{-4}

[12,18]. Indeed, if period doublings take place in a continuous-time system, the phases (time intervals between successive maxima) are varied. The phase evolution depends on the instant amplitude and one can write down $\varphi_{k+1}-\varphi_k=z_k$, where $z_{k+1}=f(z_k)$ is the Feigenbaum type mapping. If z_k contains a chaotic component, then φ_k obeys the diffusion law $\langle(\varphi_k-\varphi_0)^2\rangle\sim Dk$ and the diffusion constant D is related to the power spectrum of z $\sigma(\omega)$ at zero frequency: $D=\pi\sigma(0)$. The spectral linewidth $\Delta\omega$ is proportional to D. Taking into account the relation $\sigma_{n+1}(\omega)=2^n|\tilde{S}_n(\omega)|^2$ [16] we obtain

$$\frac{\Delta\omega_{n+1}}{\Delta\omega_n}=\frac{\sigma_{n+1}(0)}{\sigma_n(0)}=2|h^0|^2=\frac{1}{42.05...}. \tag{18}$$

The scaling relation (18) was first obtained in ref. [12].

For typical irrational frequencies we have statistical scaling with an averaged factor

$$\mu=\lim_{N\to\infty}\frac{1}{N}\sum_{l=0}^{N-1}\ln|h_l|\simeq-1.6\,. \tag{19}$$

The factor describes the averaged distance between successive new-born spectral components in the logarithmic scale [15].

Approximate expressions for the spectrum [15–18] are easily obtained from (5) if one supposes that $H_l(x)\simeq x^2$.

5. In conclusion we have developed the RG approach to scaling of the response function and the spectrum at the period-doubling transition to chaos. The peculiarity of this problem is that the renormalization transformation has a strange attractor instead of a fixed point. Similar nonattracting invariant sets have been also found in the RG treatment of invariant curve destruction [30–32]. The scaling has a statistical nature and universal constants are obtained only for averaged characteristics or exceptional periodic points.

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