SPATIAL DEVELOPMENT OF CHAOS IN NONLINEAR MEDIA

A.S. PIKOVSKY
Institute of Applied Physics, Academy of Sciences of the USSR, Gorky, USSR

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Spatial development of chaos in the media, where initial disturbances are advected (flow systems, systems with convective character of wave instability) is investigated. The problem is formulated as the study of nonlinear transformation of external perturbations, prescribed at a boundary. It is shown that the periodic field remains periodic but may be unstable with respect to secondary perturbations. Transformation of a quasiperiodic field leads to oscillations with a dense spectrum, practically undistinguishable from chaotic oscillations. In a model, where with change of a parameter convective instability turns into absolute instability, a regime of spatially period-doubling waves is found.

1. Introduction

There is a considerable interest now in the stochastic behavior of distributed systems [1,2]. Such systems often show low-dimensional behaviour, very similar to that in finite-order dynamical systems. For example, a field in a resonator may be considered as a finite set of discrete modes (higher modes damp out due to viscosity) and their evolution is described by a strange attractor (see refs. [3,4]). A different situation occurs if one considers chaos in infinite (or semi-infinite) space media. In this case the spatial spectrum is continuous, and one faces the problem of development of spatial chaos in time. Another nontrivial problem arises in the so-called flow systems, where initial disturbances are advected as they are amplified, and one has to take into account constant external forcing in order to obtain a nontrivial state in a finite spatial domain [5]. If these external disturbances are regular, the problem may be formulated as how the development of chaos takes place. In this paper we describe a mechanism of spatial development of chaos. We show also that the same mechanism works in the problem of the temporal development of spatial chaos.

2. Discrete and continuous models

Two types of systems are usually used in the investigation of chaos, that is, those with discrete and those with continuous time. A spatial variable also may be discrete or continuous. Thus, the problem of the spatial development of chaos may be formulated for different mathematical models.

2.1. Continuous time–continuous space: a complex Ginzburg–Landau equation.

This model naturally arises in weakly nonlinear studies of convectively unstable waves near the stability threshold. The complex amplitude of a spatially growing wave satisfies [3]

$$\mathrm{i}\,\frac{\partial A}{\partial x}=\mathrm{i}\,\frac{\partial k}{\partial \omega}\frac{\partial A}{\partial t}-\frac{1}{2}\frac{\partial^2 k}{\partial \omega^2}\frac{\partial^2 A}{\partial t^2}+\mathrm{i}\,\mathrm{Im}\,kA+d|A|^2A\,. \tag{1}$$

Here ω is a real frequency and k is a complex wavenumber, $\partial^2 k/\partial\omega^2$ and d are complex coefficients. With an obvious change of variables we obtain the Ginzburg–Landau equation in a non-dimensional form,

$$\frac{\partial a}{\partial x}=a+(1+\mathrm{i}c_1)\frac{\partial^2 a}{\partial t^2}+(-1+\mathrm{i}c_2)|a|^2a \tag{2}$$

(for simplicity we used the same letters for the new space and time variables).

The advantage of eq. (2) is that it describes the spatial evolution of disturbances in a semi-infinite medium, prescribed at a boundary. Thus, the statement of the problem is as follows: for $x=0$ the field at a boundary is given: $a(0, t)$, $-\infty<t<\infty$, and it is required to find the field for $x>0$.

2.2. Continuous time–discrete space: a chain of amplifiers

A chain of amplifiers without feedback is a discrete space analog of convectively growing waves (absolute instability may be modelled by a chain of oscillators [6,7]). Consider an amplifier, consisting of consequently connected nonlinear inertia-free amplifiers with transformation function $f(\ )$ and a linear lower-band filter. The chain of these amplifiers is governed by the system of ODEs,

$$\frac{\mathrm{d}u_n}{\mathrm{d}t}+u_n=f(u_{n-1})\ , \quad n=1, 2, 3, \ldots . \tag{3}$$

Here u_n is the signal at the output of the nth amplifier. We may solve the system (3) as an initial-value problem, but all initial perturbations damp out. A nontrivial state occurs only if there is an external input signal $u_0(t)$, $-\infty<t<\infty$. Thus we obtain the problem of input signal transformation (and, possibly, chaotization).

2.3. Discrete time – a mapping with diffusion

In order to obtain a discrete time model (a mapping) it is convenient to begin with a continuous time equation with nonlinear impacts:

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial x}=D\frac{\partial^2 u}{\partial x^2}+V(u)\sum_n \delta(t-nT)\ . \tag{4}$$

After solving (4) between δ-pulses we obtain the following mapping,

$$u((n+1)T, x)=\hat{L}\hat{D}f(u(nT, x))\ , \tag{5}$$

where $\hat{L}=\exp(-vT\partial/\partial x)$ is a translation operator and $\hat{D}=\exp(DT\partial^2/\partial x^2)$ is a diffusion operator. f is a nonlinear function, derived in a complex way from $V(u)$ (see ref. [8]). The boundary condition for eq. (5) may be formulated in a model form: for operator $\hat{D}$ we impose $\partial u(0)/\partial x=0$, and for operator $\hat{L}$ we suppose that $u(nT, x<0)=u_0(n)$. Note that in practical computation the variable x may be considered as discrete (see below, eq. (8)), but this is rather irrelevant, if the diffusion constant D is large (see fig. 5).

In the models (2), (3) and (5) the role of the external disturbances is more clear than in previously used chains of oscillators and coupled map lattices [2,7], so they seem to be adequate for describing convectively unstable media. Note that in the model (5) we may vary the velocity of advection by changing the parameter v, this will allow us to link a bridge (in section 5) between convectively and absolutely unstable media.

3. Evolution of periodic disturbances and their stability

We have formulated the problem of spatial development of chaos as the transformation of regular (periodic or quasiperiodic) boundary disturbances into irregular ones. If the boundary disturbances are irregular (for example, noisy), then we get the more simple problem of nonlinear noise transformation [5]. Thus, we shall restrict ourselves to the case of a regular boundary field, and will consider only the Ginzburg–Landau equation (2), because all the methods and results of this and the following section are equally applicable to the models (2), (3), and (5).

Consider first periodic boundary disturbances $a(0, t)=a(0, t+T)$. Then due to the invariance of eq. (2) with respect to time translations we get $a(x, t)=a(x, t+T)$ for all $x>0$. So for all x oscillations periodic in time are observed. In this sense there is no transition to chaos. However, variation of a field over x may have a chaotic nature (for eq. (2) such regimes with conditions periodic in t were observed in numerous computer studies, see for example ref. [3]).

Let us consider now the stability properties of this field $a(x, t)$, which is periodic in time. For a small secondary disturbance $b(x, t)$ we obtain linearization,

$$\frac{\partial b}{\partial x}=b+(1+\mathrm{i}c_1)\frac{\partial^2 b}{\partial t^2}$$
$$+(-1+\mathrm{i}c_2)(a^2b^*+2|a|^2b)\,. \qquad (6)$$

In the linear equation (6) the coefficients a^2 and $|a|^2$ are periodic functions in time with period T and their x-dependence may be irregular. Because $b(x, t)$ need not be periodic in t with period T, we may look for elementary solutions of (6) in the "Bloch form":

$$b(x,t)=\exp(\mathrm{i}\nu t)B_1(x,t)+\exp(-\mathrm{i}\nu t)B_2(x,t)\,,$$

where $B_{1,2}(x,t)=B_{1,2}(x,t+T)$ are periodic in t, and the parameter ν may be called the "quasifrequency" of a secondary perturbation. As a result, we get for $B_{1,2}$ the linear system

$$\frac{\partial B_{1,2}}{\partial x}=B_{1,2}$$
$$+(1+\mathrm{i}c_1)\left(\frac{\partial^2 B_{1,2}}{\partial t}\pm 2\mathrm{i}\nu\frac{\partial B_{1,2}}{\partial t}-\nu^2B_{1,2}\right)$$
$$+(-1+\mathrm{i}c_2)(a^2B^*_{2,1}+|a|^2B_{1,2})\,. \qquad (7)$$

For large x the solutions of (7) generally grow exponentially, $B_{1,2}\sim\exp(\lambda x)$. The exponent λ determines the stability of the secondary disturbances, and will be called the "quasi-Lyapunov exponent" because λ depends on the quasifrequency ν and for $\nu=0$ coincides with the usual Lyapunov exponent.

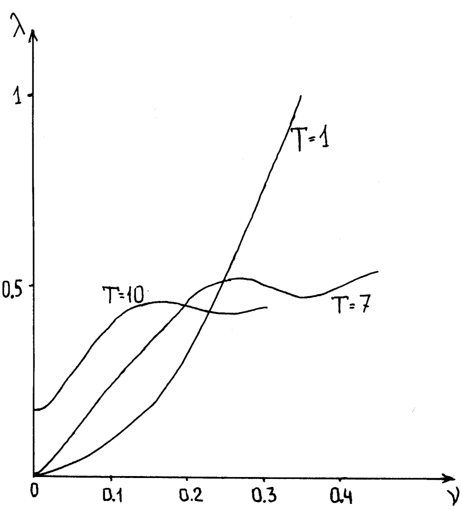

Fig. 1. Quasi-Lyapunov exponent versus quasi-frequency for stationary, periodic and chaotic states in the complex Ginzburg–Landau equation.

The quasi-Lyapunov exponent may be obtained numerically with just the same procedure as the usual Lyapunov exponent [9]. In fig. 1 the results are presented for the case $c_1=3$, $c_2=5$, and for different periods of the initial disturbance T. For $T=1$ in (2) a steady (in x) state occurs, for $T=7$ a periodic one and for $T=10$ a chaotic one. All these regimes are unstable: the secondary disturbances with periods different from T, grow with x. The result of this instability depends on the particular type of secondary disturbances. For noisy disturbances the spatial growth of noise may take place leading at large x to spatial–temporal turbulence (in the same way as for noisy initial disturbances [5]). The more interesting situation of regular–periodic–secondary disturbance is thoroughly investigated below.

4. Spatial development of quasiperiodic disturbances

The case of periodic secondary disturbances may be treated as follows. Consider eq. (2) with the boundary field $a(0,t)=a_0(t)+\epsilon a_1(t)$ where $a_0(t)=a_0(t+T_0)$ and $a_1(t)=a_1(t+T_1)$ are periodic functions with incommensurate frequencies $\omega_0=2\pi/T_0$ and $\omega_1=2\pi/T_1$. Qualitatively, the spatial development of this field proceeds as follows. For small x a regime periodic in t develops. Then due to the secondary instability described above the disturbance $a_1(x,t)$ begins to grow. During this growth various combinational spectral components $m\omega_0+n\omega_1$ appear. These components also grow due to instability. As a result we may obtain for large x a dense, almost continuous spectrum.

We have modelled this process numerically. Eq. (2) was solved with an implicit finite-difference method. Because we cannot solve the equation in an infinite domain, conditions periodic in t were used. Thus the frequencies are commensurate: $\omega_0/\omega_1=p/q$, but for large p and q we may achieve a good approximation for an irrational number, so the development of the quasiperiodic regime will be modelled rather well. We approximated the "golden mean" irrational number $(\sqrt{5}+1)/2$ by the ratio $p/q=233/144$. The obtained spatial development of the temporal spectrum for a boundary field with frequencies

$\omega_1 = \pi/10$ and $\omega_0 = \omega_1 \times 233/144$ is presented in fig. 2. One can see how the spectrum becomes more and more dense, and this may be interpreted as the chaotization of the wave field. This situation seems to be paradoxical: rigorously, the spectrum is discrete (and the process is quasiperiodic) but it looks like a continuous one. In order to clarify the paradox let us consider some characterizations of the quasiperiodic state.

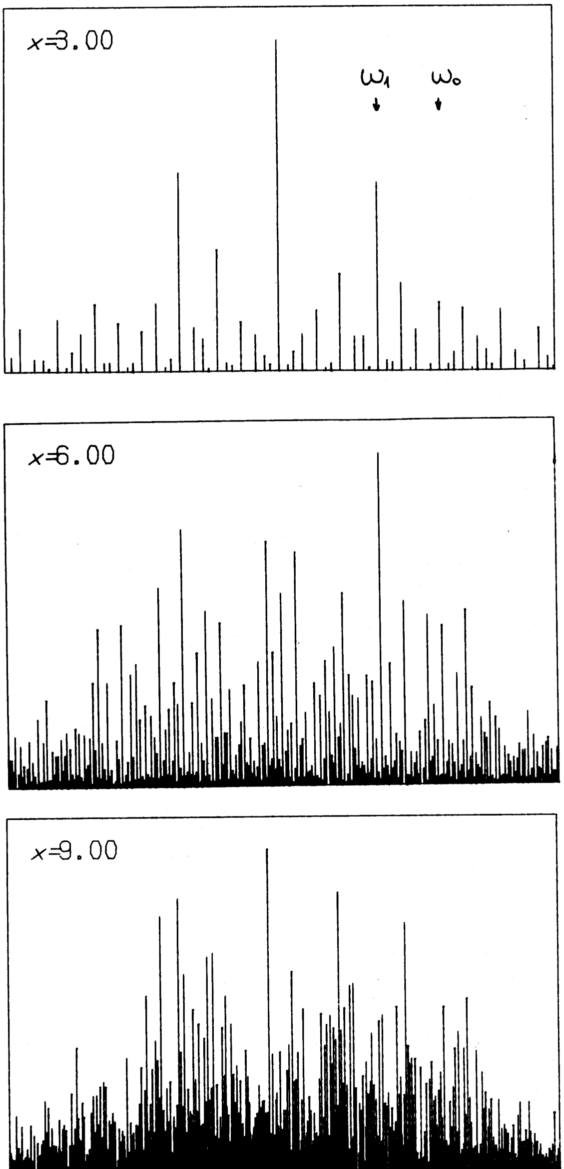

Fig. 2. Spatial development of the temporal spectrum of the quasiperiodic field.

The first approach is in the interpretation of quasiperiodic oscillations as an object in effective phase space. We construct this in the following way: form a sequence of real numbers $u_l = |a(x, lT_0)|^2$, $l=0, 1, 2, ...$ and plot these points in the "phase plane" (u_l, u_{l+1}). If we plot an infinite number of points $0 \leqslant l < \infty$ we shall obtain a closed curve – a section of a two-dimensional torus. In our numerical modelling we obtain a finite set of points, which approximate this curve. The results of such data processing are presented in fig. 3. One can see that for large x the torus becomes more folded and tangled, and this kneading, interwoven line cannot be practically distinguished from the projection of a high-dimensional strange attractor.

Another approach to the characterization of a quasiperiodic regime is in the construction of a generating function ($g(y)$ [10] which is connected with the quantities u_l in the following way: $u_l = g(lp)$, where $p = \omega_1/\omega_0$, and $g(y) = g(y+1)$ is periodic with period 1. This generating function is presented in fig. 4. With the growth of x the function becomes more and more indented (but remains smooth), which corresponds to the appearance of new spectral components.

In just the same way the spatial development of a quasiperiodic input signal takes place in a chain of nonlinear amplifiers (3). The system (3) is a system of ODEs and all ordinary Lyapunov exponents (which determine the temporal growth of the disturbances) here are equal to -1. However, the quasi-Lyapunov exponent, describing the spatial development of the disturbances, may be positive. For a quasiperiodic input signal we have a two-dimensional torus in phase space (two zero Lyapunov exponents arise from the input signal) but the view of this torus is different for different n: for large n it is practically indistinguishable from a strange attractor [#1].

In the discrete model (5) we also obtained for large v similar regimes of spatial chaotization of a quasi-

[#1] In refs. [11,12] strange but nonchaotic attractors were found in a quasiperiodically forced first-order nonlinear system.

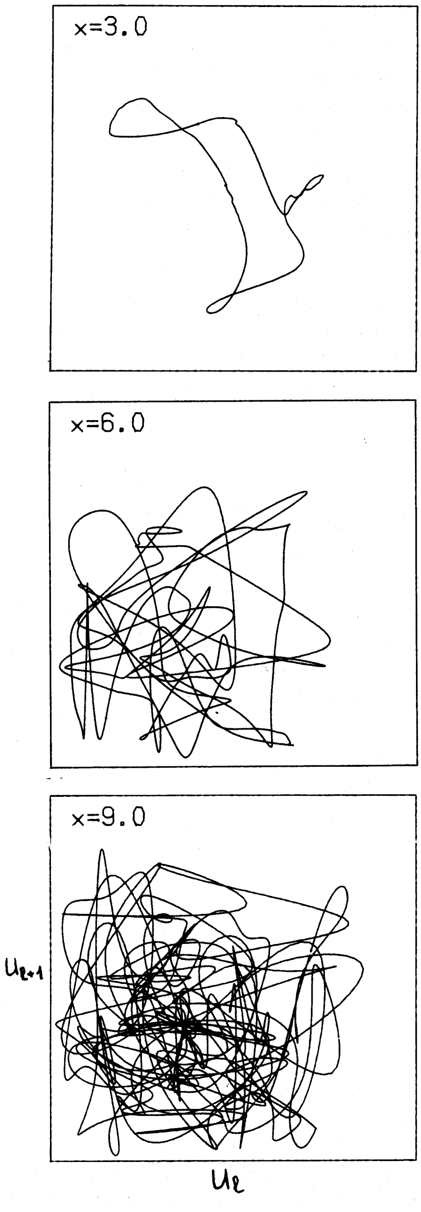

Fig. 3. Cross-sections of the two-dimensional torus in effective phase space.

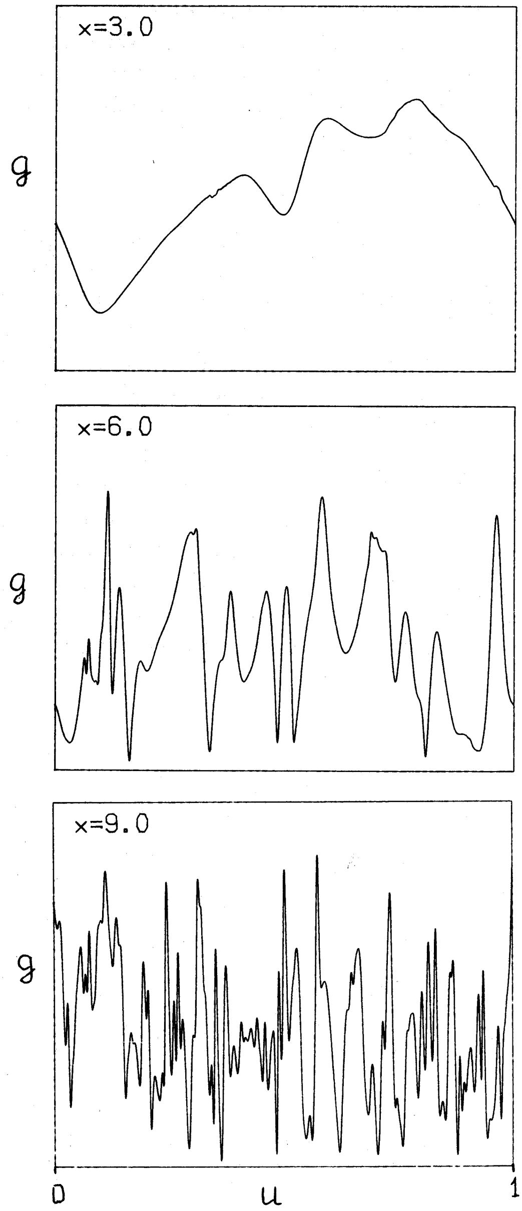

Fig. 4. Evolution of the generating function.

periodic boundary field. However, for small values of the "flow velocity" v, when the convective instability turns into absolute instability, we obtained another type of transition to chaos –through spatial period-doublings.

5. Transition to chaos through spatial period-doublings

In numerical experiments it is convenient to use a discrete-space version of the mapping (5). The mapping describes the evolution of a field $u(n, x)$

transformed to the problem described above of the spatial development of temporal chaos. Indeed, unstable waves in an infinite medium may be governed by the complex Ginzburg–Landau equation in the "dual" form [3],

$$\frac{\partial a}{\partial t}=a+(1+\mathrm{i}\bar{c}_1)\frac{\partial^2 a}{\partial x^2}+(-1+\mathrm{i}\bar{c}_2)|a|^2 a\,.$$

We may state the problem as follows: an initial field $a(x,0)$ is given and we have to investigate how spatial complexity develops. This dual form is completely analogous to (2), if one changes $x\leftrightarrow t$, $\omega\leftrightarrow k$. Thus, all the results of sections 3,4 may be directly applied to this problem: the periodic (in space) initial field remains periodic but may be unstable, the quasiperiodic initial field leads to an almost chaotic spatial state with a dense spectrum like fig. 2.

It should be noted that the same phenomenon may be observed for conservative (of course, non-integrable) equations. For example, we obtained pictures like figs. 1–4 for a discrete φ^4 model

$$\frac{d^2\varphi_i}{dt^2}-\varphi_i+\varphi_i^3-D(\varphi_{i-1}-2\varphi_i+\varphi_{i+1})=0\,,$$

$$i=\text{integer}\,.$$

The same approach holds for discrete-time mappings with diffusion,

$$u_{n+1}(x)=\hat{D}f(u_n(x))\,,$$

considered in an infinite spatial domain. This problem may be considered as a spatial–temporal "duality" of the model (3).

7. Conclusion

We described the mechanism for spatial development of chaos in a flow system, where disturbances move in one direction. The peculiarity of this situation is that here we deal not with a dynamical system in the usual sense, but with a nonlinear transformation of external disturbances. A nontrivial regime is obtained if these disturbances are quasiperiodic – then an image of "quasi-chaos" is a folded torus in an effective phase space. When taking into account the possibility of absolutely unstable modes, we observed a regime of spatial period-doublings.

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