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Randomization of a Signal in a Chain of Nonlinear Amplifiers*

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The transformation of a regular input signal into a random signal in a chain of nonlinear amplifiers is studied theoretically and numerically. Nonlinear amplifiers with zero time constant, a square-law characteristic, and a low-pass filter are studied as a specific example. It is shown that the appearance of chaos can be explained as the result of the development of secondary instabilities. The appearance of chaos in a quasiperiodic signal is studied in detail.

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INTRODUCTION

In recent years there has been great interest in radio electronic noise generators, operating on the principle of stochastic self-excitation [1-4]. In spite of the fact that the operating states and statistical characteristics of the output signal of different schemes differ significantly, there are also a number of common features: the existence of a strongly nonlinear element, a linear circuit employed as a filter, and feedback. If one starts from such a generalized representation, then the mechanism of the generation of noise-like signals consists of the following: the nonlinear element "knocks off" the spectral components and in this manner broadens the spectrum of oscillations, the linear filter limits the spectral band, shifts the phase, and in so doing prevents synchronization, while the positive feedback ensures that the entire process is stationary. This representation implies that the mechanism of the occurrence of chaos is not related directly with the feedback. It is therefore possible to construct non-self-excited devices which produce chaos in the signal.

In this paper the transformation of a regular input signal into a chaotic signal in a chain of nonlinear amplifiers is studied theoretically and numerically [5]. Nonlinear amplifiers with zero time constant, a quadratic charactristic, and a low-pass filter are studied as a specific example. The occurrence of chaos can be represented as the result of the development of secondary instabilities. The occurrence of chaos in a quasiperiodic signal is followed in detail; the fractal torus is the mathematical image of the chaos in this case.

1. BASIC EQUATIONS

Consider a chain of amplifiers, each of which consists of an amplifier A, with zero time constant, and a filter F connected in series. Let $u_n(t)$ be the signal at the output of the n-th section. Assume that the amplifier has a zero time constant, i.e., the signal at its output $u_n^o(t)$ can be written algebraically in terms of the signal at the input $u_{n-1}(t)$:

$$u_n^o(t)=f(u_{n-1}(t)). \tag{1}$$

The linear filter is described by the operator $\hat{L}$:

$$u_n(t)=\hat{L}u_n^o(t). \tag{2}$$

Combining (1) and (2) we obtain a relation between the signal at the output of the $(n-1)$-th and

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n-th elements:

$$u_n(t)=\hat{L}f(u_{n-1}(t)), \tag{3}$$

where the input signal $u_0(t)$ must be given.

We shall study below as a specific example a system in which the amplifier is described by the quadratic function

$$f(u)=4u(1-u), \tag{4}$$

while the linear filter is a low-pass filter with the frequency character

$$L(\omega)=1/(1+i\omega) \tag{5}$$

(here the time constant of the filter is taken as the unit of time). In this case system (3) can be written as a system of ordinary differential equations:

$$\frac{du_n}{dt}+u_n=f(u_{n-1}), \quad n=1,2,3,\ldots . \tag{6}$$

2. TRANSFORMATION OF A PERIODIC SIGNAL

We shall study the case of a periodic input signal $u_0(t)=u_0(t+T)$. Then, as follows from the uniformity of Eqs. (3) in time, the period of the signal $u_n(t)$ equals T for any n: $u_n(t) = u_n(t + T)$. Thus chaos in the signal does not occur in the usual sense. Here, however, the regime of "spatial stochastization," when the signal, while remaining periodic in time, varies irregularly as a function of the variable n, is possible. The mechanism for the appearance of such a regime can be understood by studying a system without a filter. Using (3), taking (4) into account, we obtain

$$u_{n+1}(t)=f(u_n(t))=4u_n(t)(1-u_n(t)). \tag{7}$$

The transformation (7), as is well known (6), leads to a chaotic variation of u with respect to n. The variable in (7) is the discrete time, while the real time t is a parameter. The values of $u_n(t)$ and $u_n(t')$ for any t and t' transform independently, as a result of which the signal $u_n(t)$ becomes distorted, becoming enriched in harmonics, but remains periodic. The linear filter smoothes the signal, and its effect is all the stronger the shorter the period T. For $T \ll 1$ all harmonics of the input signal except the zeroth harmonic are damped as a function of n, and the regime of chaotic variation of the constant signal with respect to n is established.

When the period T changes the following changes occur. For $T = T_1$ the constant signal regime becomes unstable. The threshold of the instability can be found by the method described in [7]. Consider a small harmonic perturbation $\delta u_n=c_n\exp(it2\pi/T)$ against the background of a constant signal u_n°. Then, linearizing (3), taking into account (4) and (5), we obtain for the amplitude of the perturbation

$$c_n=L\left(\frac{2\pi}{T}\right)f'(u_{n-1}^{0})c_{n-1}=L\left(\frac{2\pi}{T}\right)4(1-8u_{n-1}^{0})c_{n-1}$$

or for the variable $z_n=\ln|c_n|$

$$z_n=\ln\left|L\left(\frac{2\pi}{T}\right)\right|+\ln|4(1-8u_{n-1})|+z_{n-1}. \tag{8}$$

Averaging (8) and using the well-known [6] expression $\langle\ln|4(1-8u_{n-1}^{0})|\rangle=\ln 2$, we obtain from the condition $\langle z_n\rangle=\langle z_{n-1}\rangle$ the threshold of the instability

$$T_1=\pi\sqrt{4/3}=3.627\ldots .$$

The dynamics of the regimes that occur for $T > T_1$ is as follows. For $T_1 < T < T_2 \simeq 6.5$ a chaoti regime as a function of n is observed; for $T_2<T<T_3\simeq 7.9$ a regular regime (periodic or quasiperiodic) is observed; and, for $T > T_3$ chaos is observed.

We shall examine the instability of the transformation in the chain of amplifiers of a signal periodic as a function of t. Let the input signal have the form

$$u_0(t)=a(t)+\varepsilon b(t), \tag{9}$$

where $a(t)=a(t+T)$ is a periodic function, $\varepsilon \ll 1$, and $b(t)$ is a perturbation, having in the general case spectral components that are not commensurate with $2\pi/T$. Linearizing (3) against the background of the solution $v_n(t)$, corresponding to $u_0(t) = a(t)$, we obtain

$$b_n(t)=\hat{L}f'(v_{n-1}(t))\,b_{n-1}(t). \tag{10}$$

Equation (10) defines a linear mapping, periodic as a function of t, in function space. We shall seek its solution in the form

$$b_n(t)=e^{i\nu t}c_n(t), \tag{11}$$

where $c_n(t)=c_n(t+T)$. This representation is analogous to Bloch's representation, employed in the solution of linear differential equations with periodic coefficients. Henceforth, the parameter ν will be termed the quasifrequency. Using (11), Eq. (10) can be written in the form

$$c_n(t)=\hat{L}(\nu)f'(v_{n-1}(t))\,c_{n-1}(t), \tag{12}$$

where $\hat{L}(\nu)=e^{-i\nu t}\hat{L}e^{i\nu t}$. The mapping (12), unlike (10), now operates on the class of functions periodic in t, but depends on the quasifrequency, as a parameter, and in the general case it depends chaotically on n. For this reason it makes sense to talk about the statistical properties of the solution. In the general case for large values of n the solution of (12) behaves asymptotically as

$$C_n(t)\sim e^{\lambda n}, \tag{13}$$

where λ is the index of stability, henceforth termed the quasi-Lyapunov index. Relation (13) must be understood in the statistical sense: on the average the signal $c_n(t)$ increases from amplifier to amplifier with the index λ. For $\nu = 0$, Eq. (12) describes the evolution of perturbations with the same period as the main signal. In this case λ is the standard Lyapunov index (with respect to the discrete "time" n), which determines whether or not chaos will occur [8]. In the general case λ depends on the quasifrequency ν and determines the stability of the periodic signal: $\lambda(\nu)>0$ means that a perturbation of the form (11) increases as a function of n), i.e., $\lambda(0)>0$, the existence of unstable quasifrequencies follows from the continuity of the dependence of the index λ on the quasifrequency ν. The physical mechanism of this instability consists of the following. Sections of the signal, faraway as a function of t, interact weakly, so that by virtue of the chaotic instability their small perturbations grow as a function of n.

The quasi-Lyapunov indices can be determined numerically by the standard method for finding the Lyapunov index [6]. The results of calculations based on Eqs. (3)-(5) are presented in Fig. 1. The characteristic values of λ in Fig. 1 ($\lambda \simeq 0.1...0.7$) show that the perturbation increases significantly on 10-20 elements.

3. THE APPEARANCE OF CHAOS IN THE SIGNAL

The instability of the transformation of a periodic signal described above enables us to represent the qualitative picture of the occurrence of chaos in the signal in a chain of amplifiers. Let the signal at the input have the form (9), where $a(t)$ is a periodic signal, while $b(t)$ is a noise or periodic signal with incommensurate period. If $b(t)$ is a noise signal, its spectral components increase owing to the instability, as a result of which for large values of n a chaotic process whose statistical characteristics are independent of n forms (they are described in greater detail below).

We shall examine in greater detail the case when the input signal is quasiperiodic, i.e., the frequencies ω_1 and ω_2 of the components a and b are not commensurate. Under a nonlinear transformation spectral components appear at the difference and sum frequencies, which grow by virtue of the described instability; new spectral components appear, etc. As a result the spectrum becomes indistinguishable from a continuous spectrum, and the signal can be regarded as random. Chaos appears in an analogous manner for a multifrequency input signal.

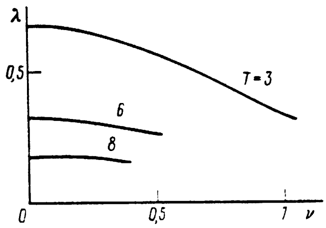

Fig. 1

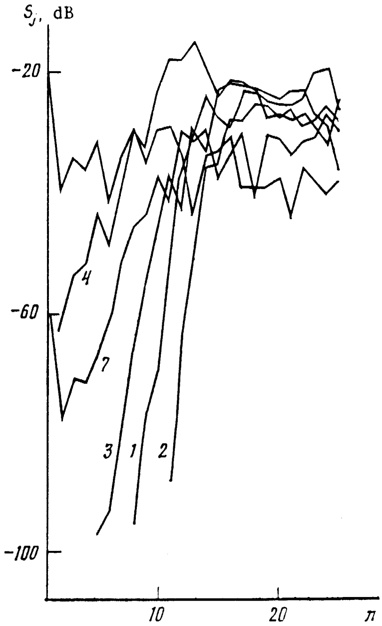

Fig. 2

Fig. 1. Quasi-Lyapunov indicators for regimes with different periods T.

Fig. 2. Growth of the amplitudes S of the spectral components with a two-frequency input signal. The numbers on the curves correspond to harmonics of the fundamental frequency $\Omega = 2\pi/T_m$.

The numerical method is very important in the computer simulation of the appearance of chaos in the signal. Seeds on unperturbed frequencies must not appear. The spectral computational method satisfies this condition; in this method the evolution of the spectral components of the signal as a function of n is followed: the action of the filter reduces to multiplication by $L(\omega)$, while the action of the nonlinear transformation (5) reduces to a convolution*. On discretization one actually must deal with a process periodic in t (and, of course, with commensurate frequencies); growth of the spectral components leads not to chaos, but rather to a periodic signal with a long period.

Figure 2 shows the results of calculations of this transformation, in which the maximum period of the signal $T_m = 50$. A component with a period $T_2 = T_m/7$ at a level of -60 dB was added against the background of a regime chaotic in n with $T_1 = T_m/11$. As one can from the figure, the combination frequencies grow very rapidly and this growth virtually stops by $n \simeq 20$.

We shall discuss the mathematical nature of the above chaos, arising with a two-frequency input signal. The chain of n amplifiers is described by a system of n ordinary differential equations (6). For this system it is possible to find Lyapunov indices, describing the stability of the motions as a function of t. It is obvious from system (6) that two indicators equal zero (they correspond to the two phases of the input signal), while the remaining indicators are negative and equal -1. Thus from the mathematical viewpoint for any value of n the attractor is a two-dimensional torus. This is inconsistent with the observed noise-like nature of the signal. The paradox is explained by the fact that as n increases the torus becomes increasingly irregular and in the limit as $n \to \infty$ it becomes fractal (Fig. 3). In the spectral language the existence of a torus indicates that the spectrum contains only frequencies of the

*Note that the use of a double fast Fourier transform instead of a convolution leads to the appearance of small values of all spectral components, and as a result of instability these seeds grow rapidly.

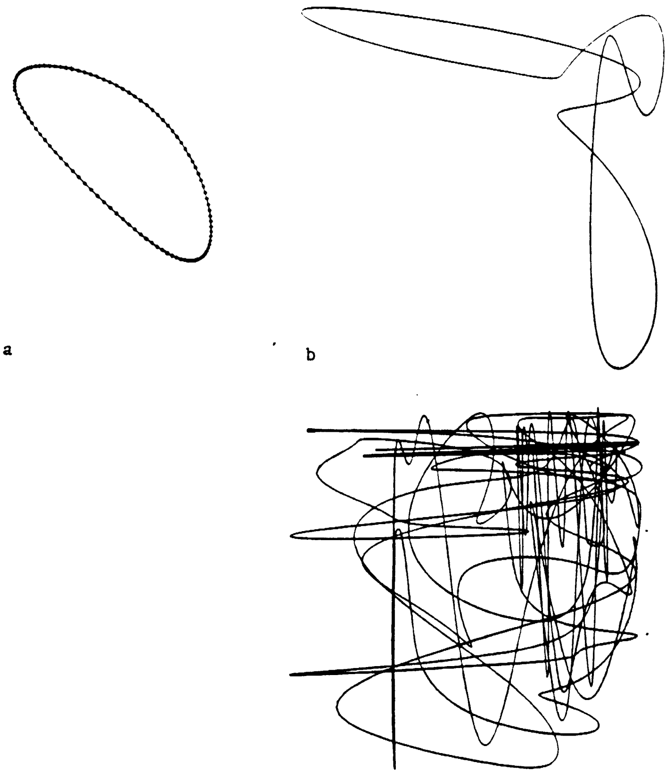

Fig. 3. Evolution of a two-dimensional torus along a chain of amplifiers for a two-frequency input signal with $T_1 = 10$, $T_2 = 5(1 + \sqrt{5})$ and $n = 1$ (a), 5 (b), 10 (c). The dots in the plane $\{u_k, u_{k+1}\}$, where $u_k = u(kT_1)$, represent the section of the torus.

type $k\omega_1 + m\omega_2$. The fractalization of the torus, however, is due to the fact that the combination harmonics with large values of k and m are of the same order of magnitude as harmonics with small k and m. Therefore the mathematical representation of chaos in a chain of amplifiers with a two-frequency (l-frequency) perturbation is a fractal two-dimensional (l-dimensional) torus.

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