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NONLINEAR INTERACTION OF WAVES IN AN INHOMOGENEOUS MEDIUM

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Nonlinear resonant interaction of three waves in a dispersive medium is considered within the framework of Hamiltonian formalism. The steady-state distribution function is found for random inhomogeneity and simplified diffusion equations are derived. Interaction in a periodically inhomogeneous medium is defined by the character of the inhomogeneity. For fine scale inhomogeneity the interaction slows significantly, while for a coarse scale it becomes chaotic. The dependence of the chaos level on the Manly–Row integrals is studied.

Nonlinear resonant interaction of waves plays a major role in hydrodynamics, nonlinear optics, and plasma physics. One of the fundamental elementary processes involved is energy exchange between three resonantly coupled waves in a dispersive medium. Inhomogeneity of the medium has a significant effect on the interaction, leading to reduction of the wave phases. Of greatest interest are the cases of relatively smooth random [1, 2] and periodic [3-5] inhomogeneities. In the present study we will use a unified Hamiltonian approach to consider nondegenerate three-wave interaction in an inhomogeneous medium. The equations of wave dynamics will be derived in Hamiltonian form in Sec. 2. Section 3 will consider a randomly inhomogeneous medium. To deal with this problem Abramovich [1] defined an amplitude and phase distribution function for zero mean detuning. In the present study the distribution function will be found for any detuning. Moreover, a simplified diffusion equation will be derived for the case of large inhomogeneity and large detuning. Section 4 will consider a periodically inhomogeneous medium. For the case of fine scale (as compared to the characteristic nonlinear interaction length) inhomogeneities an averaging method analogous to that of [4, 5] will be used. If the scales of the inhomogeneity and the nonlinearity are of the same order of magnitude, the chaotic regime of energy exchange is possible. The most intense stochastization occurs in a situation close to degenerate, where the amplitudes of low-frequency waves may vanish.

1. Hamiltonian Form of the Equations. The steady-state process of resonant wave interaction can be described by abbreviated equations for the complex amplitudes a_j ($j = 1, 2, 3$) [6]:

$$da_1/dz = -i\beta_1 a_2 a_3 \exp[-i\psi(z)],$$
$$da_{2,3}/dz = -i\beta_{2,3} a_1 a^*_{3,2} \exp[-i\psi(z)],$$

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$$\psi(z) = \int^{z} \Delta k(z')dz'. \qquad (1)$$

Here $\psi(z)$ is the phase difference, determined by the local detuning of the wave vectors $\Delta k = k_1 - k_2 - k_3$, β_j are constant interaction coefficients. We transform to intensities I_j and phases φ_j:

$$a_j = -i\beta_j^{1/2} I_j^{1/2} \exp[i(\varphi_j + \psi)]$$

and the coordinate $x = z(\beta_1\beta_2\beta_3)^{1/2}$. Then Eq. (1) reduces to a Hamiltonian system with Hamiltonian

$$H_1 = 2(I_1 I_2 I_3)^{1/2} \sin(\varphi_1 - \varphi_2 - \varphi_3) - \Delta(x)(I_1 + I_2 + I_3),$$

where $\Delta = \Delta k(\beta_1\beta_2\beta_3)^{-1/2}$. We perform a canonical transformation to the variable J_j, θ_j with generating function

$$F = (\varphi_1 - \varphi_2 - \varphi_3)J_1 + \varphi_2 J_2 + \varphi_3 J_3,$$

obtaining the Hamiltonian

$$H_2 = -2(J_1(J_2 - J_1)(J_3 - J_1))^{1/2} \sin\theta_1 - \Delta(J_2 + J_3 - J_1).$$

The variables J_j, θ_j are related to I_j, φ_j by the expressions

$$J_1 = I_1,\ J_2 = I_1 + I_2,\ J_3 = I_1 + I_3,\ \theta_1 = \varphi_1 - \varphi_2 - \varphi_3,\quad \theta_2 = -\varphi_2,\quad \theta_3 = -\varphi_3.$$

Since the Hamiltonian H_2 is independent of θ_2 and θ_3, J_2 and J_3 are integrals of motion (Manly–Row integrals). Therefore the constant part of the Hamiltonian can be dropped. Denoting $J = J_1$, $\theta = \theta_1$, we finally obtain the Hamiltonian system

$$dJ/dx = -\partial H/\partial\theta,\quad d\theta/dx = \partial H/\partial J,$$
$$H = -2\sin\theta\,[J(J_2 - J)(J_3 - J)]^{1/2} + \Delta(x)J. \qquad (2)$$

It follows from Eq. (2) that the characteristic nonlinear interaction length $L_{nl} \sim J_{2,3}^{-1/2}$.

2. Randomly Inhomogeneous Medium. We will assume that the detuning $\Delta(x)$ has the form $\Delta(x) = \Delta_0 + \Delta_1(x)$, where Δ_0 is constant and Δ_1 is variable with a zero average. If the characteristic size of the inhomogeneity is small in comparison to L_{nl}, then the quantity $\Delta_1(x)$ can be approximated by a Gaussian δ-correlation process

$$\langle \Delta_1(x)\Delta_1(x')\rangle = 2D\delta(x - x').$$

The parameter $D^{-1} = L_s$ has the sense of the characteristic length for multiple wave scattering in the inhomogeneous medium [1]. This approximation permits the use of standard methods to transform to a Fokker–Planck equation for the probability density $W(J, \theta)$:

$$\frac{\partial W}{\partial x} = 2\cos\theta\, h(J)\frac{\partial W}{\partial J} + (\Delta_0 - 2\sin\theta h'(J))\frac{\partial W}{\partial \theta} + D\frac{\partial^2 W}{\partial \theta^2}, \qquad (3)$$

where the notation $h(J) = [J(J_2 - J)(J_3 - J)]^{1/2}$ has been introduced. Equation (3) has the obvious steady-state solution

$$W = \text{const},\quad 0 < J < \min(J_2, J_3),\quad 0 \leqslant \theta < 2\pi, \qquad (4)$$

corresponding to a microcanonical distribution. We note that the uniform distribution of Eq. (4) was obtained in [1] for the special case $\Delta_0 = 0$.

A paradoxical conclusion follows from Eq. (4): the limiting steady-state distribution defining the efficiency of wave interaction does not depend on the detuning constant Δ_0. In fact, the time required for establishment of this steady state does depend significantly on detuning.

We will consider in greater detail the case of large detuning constants. We introduce the small parameter $L_p = \Delta_0^{-1}$, having the sense of the phase reduction length. If $L_p \ll L_{nl}$, L_s, the standard perturbation method for Hamiltonian systems with rapidly rotating phase can be used [7]. Transforming to new variables $\bar{J}$, $\bar{\theta}$ defined by

$$J = \bar{J} - 2L_p \sin\bar{\theta}\, h(\bar{J}),\quad \theta = \bar{\theta} - 2L_p \cos\bar{\theta}\, h'(\bar{J}),$$

we obtain a Hamiltonian

$$\bar{H} = \Delta_0\bar{J} + \Delta_1(x)(\bar{J} - 2L_p \sin\bar{\theta}\, h(\bar{J})). \qquad (5)$$

From Eq. (5) we transform to the Fokker–Planck equation for the probability density $\bar{W}(\bar{J}, \bar{\theta})$, which after averaging over the rapidly rotating phase $\bar{\theta}$ has the form

$$\frac{\partial \bar{W}}{\partial x} - \Delta_0 \frac{\partial \bar{W}}{\partial \bar{\theta}} = \frac{\partial}{\partial \bar{J}}\left[2DL_p^2 h^2 \frac{\partial \bar{W}}{\partial J}\right] + D\frac{\partial^2 \bar{W}}{\partial \bar{\theta}^2}. \tag{6}$$

It is evident from Eq. (6) that the phase diffusion coefficient is significantly higher than the intensity diffusion coefficient. Therefore, it can be assumed that $\bar{W}(\bar{J}, \bar{\theta}) \approx (1/2\pi)P(\bar{J})$, and for $P(\bar{J})$ we obtain

$$\frac{\partial P}{\partial x} = \frac{2D}{\Delta_0^2}\frac{\partial}{\partial \bar{J}}\left(h^2(\bar{J})\frac{\partial P}{\partial \bar{J}}\right). \tag{7}$$

It follows from Eq. (7) that the length required for establishing a steady-state distribution P = const for the case of large detunings is of the order of $L_s L_{nl}^2 L_p^{-2}$.

In the case of a very intense fluctuation component in the inhomogeneity, where $L_s \ll L_{nl}$, L_p, one can also obtain a simplified diffusion equation from Eq. (3). We represent the distribution function by a Fourier series

$$W(J, \theta, x) = \sum_m W_m(J, x)\exp(im\theta)$$

and substitute in Eq. (3). We then obtain

$$\frac{\partial W_m}{\partial x} + h(J)\left[\frac{\partial}{\partial J}(W_{m-1} + W_{m+1})\right] + i\Delta_0 m W_m - h'(J)[(m-1)W_{m-1} + (m+1)W_{m+1}] = -m^2 D W_m.$$

Since D is a large parameter we can assume that only W_0, $W_{\pm 1}$ are nonzero (i.e., the distribution function over θ is close to uniform). As a result we obtain a closed equation for W_0:

$$\frac{\partial W_0}{\partial x} = \frac{2D}{D^2 + \Delta_0^2}\frac{\partial}{\partial J}\left(h^2(J)\frac{\partial W_0}{\partial J}\right). \tag{8}$$

We note that for $\Delta_0 \gg D$ Eq. (8) transforms to Eq. (7), so that it can be used for both $L_s \gg L_{nl}$, L_p and $L_p \gg L_{nl}$, L_s. From Eq. (8) it is simple to obtain equations for the moments $N_p = \langle I^p \rangle$:

$$\frac{dN_p}{dx} = \frac{2D}{D^2 + \Delta_0^2}\, p[3N_{p+1} - 2(I_2 + I_3)N_p + I_2 I_3 N_{p-1}].$$

In particular, for p = 1 we have

$$\frac{dN_1}{dx} = \frac{2D}{D^2 + \Delta_0^2}[I_2 I_3 - N_1 I_2 - N_1 I_3 + 3(N_2 - N_1^2)].$$

Equation (9) differs from the expression normally obtained in the chaotic phase approximation only in the last term $N_2 - N_1^2$. This term is small if the distribution function W_0 is close to a δ-function. In fact, this means that in the chaotic phase approximation a set of systems with identical wave amplitudes is used.

3. Periodically Inhomogeneous Medium. In a periodically inhomogeneous medium the detuning has the form $\Delta = \Delta_0 + \Delta_1(x)$, where $\Delta_1(x)$ is a periodic function. For definiteness we will assume the inhomogeneity sinusoidal: $\Delta_1 = \rho\cos\varkappa x$.

If the period of the inhomogeneity $L_i \approx \varkappa^{-1}$ is small ($L_i \ll L_{nl}$), then the equations of wave interaction can be averaged over this period. To do this we first transform to the variable $\alpha = \theta - \rho\cos\varkappa x$, for which Hamiltonian (2) takes on the form

$$H(J, \alpha, x) = -2\sin(\alpha + \rho\varkappa^{-1}\sin\varkappa x)h(J) + \Delta_0 J.$$

Averaging the Hamiltonian by the standard method of [8], we obtain

$$\bar{H}(J, \alpha) = -2J_0(\rho\varkappa^{-1})\sin\alpha\, h(J) + \Delta_0 J,$$

where $J_0(\rho\varkappa^{-1})$ is a zeroth-order Bessel function. Thus, in a medium with fine inhomogeneities energy exchange between waves may slow significantly, and at a certain relationship between the period and the magnitude of the inhomogeneity it may in general cease entirely in the first approximation.

In the case where the period of the inhomogeneity is of the order of magnitude of the nonlinear interaction length ($L_i \approx L_{nl}$), resonance is possible. In light of the nonlinearity

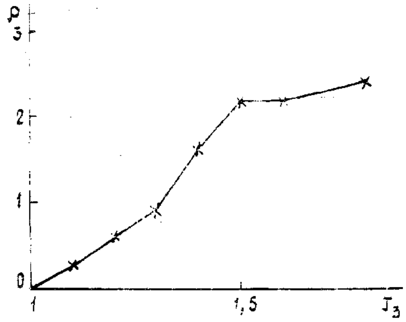

Fig. 1

of system (2), for sufficiently large ρ the oscillations may become stochastic [7]. The value of ρ at which chaos develops can be estimated from the following considerations. For simplicity we will take $\Delta_0 = 0$. In the absence of inhomogeneity ($\rho = 0$) the solution of Eq. (2) can be expressed in terms of elliptical functions [9]. The degree of nonlinearity, equal to the derivative of frequency with respect to amplitude, is maximal for the solution corresponding to a zero value of the Hamiltonian. For this solution $L_{nl} = J_3^{-1/2} K(J_2 J_3^{-1})$, where K is an elliptical integral of the first kind (for definiteness we assume $J_2 \leq J_3$). From this it is evident that since $K'(0) = 0$, $K'(1) = \infty$, the maximum sensitivity to periodic action will occur at $J_2 = J_3$, while for $J_2 \ll J_3$ the degree of nonlinearity is small and oscillations will become stochastic only at large ρ. Figure 1 shows results of numerical modeling of Eq. (2), which confirm this conclusion. Shown here are values of ρ, at which the solution becomes chaotic with initial conditions $\theta = \pi$, $J = J_2/2$ at $L_i = 3$, $J_2 = 1$, as a function of J_3. It is evident that as $J_3 \to 1$ stochastization develops even at small ρ, which is related to the existence in the degenerate system ($J_2 = J_3$) of a homoclinic trajectory [4, 5].

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