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INFLUENCE OF NOISE ON THE STATISTICS OF RANDOM SELF-OSCILLATIONS

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The influence of noise on the statistics of a strange attractor is analyzed on the example of a simple self-excited generator of radio-range noise. It is shown that to calculate the macroscopic statistical characteristics one must, first, find the invariant distribution function and, second, solve the statistical problem of escape from the region. The corresponding corrections in the case of low noise are found analytically.

1. Random self-oscillations are observed in many nonlinear systems. The strange attractor serves as a mathematical form of them [1]. In actual situations, however, fluctuations (noise) are always present, so that rigorously determinate models are not fully adequate. The influence of noise is especially strong at points of transition from regular to random behavior (see [2]). But if developed random oscillations occur, low noise does not result in qualitative changes (although it does destroy deterministic predictability). Here the following statement of the problem seems natural: To determine how the statistical characteristics of random oscillations vary under the action of noise. The difficulty in solving this problem consists in the fact that even the statistical characteristics of a purely determinate, random regime can be determined analytically in few situations. One of the simplest systems for which an analytic description is possible is the self-excited generator of radio-range noise proposed in [3]. The statistical characteristics of self-oscillations in this generator were determined analytically in [4, 5]. The influence of noise on the statistics of the signal in a noise generator is considered in the present paper. This problem was the subject of [6], but it was not fully solved there. The basic equations describing the oscillations in a self-excited generator in the presence of noise are derived in Sec. 2 of the present paper. In Sec. 3 the statistical characteristics of the signal are found in general form. The case of low noise is treated in Sec. 4.

2. A diagram of a self-excited generator of radio-range noise is presented in Fig. 1a. The only nonlinear element here is a tunnel diode, the characteristic curve of which is shown in Fig. 1b. The operation of the generator is described by the equations

$$\dot{x} = 2hx + y - \sigma z + \eta_1(t), \quad \dot{y} = -x + \eta_2(t),$$
$$\varepsilon \dot{z} = x - f(z) + \eta_3(t). \tag{1}$$

Here x, y, and z are dimensionless variables proportional to the current I and the voltages U and V, respectively; $\varepsilon \ll 1$, σ, and h are parameters [3], and $\eta_i(t)$ is the external noise.

In the absence of noise, the small parameter ε enables us to separate the motions into fast and slow and to reduce the problem to the one-dimensional mapping of following. This mapping, connecting successive maxima of the quantity y, is presented in Fig. 2. This mapping is written out analytically for a piecewise-linear approximation of the diode characteristic curve [5], but the corresponding equations have a cumbersome form. Therefore, we confine our-

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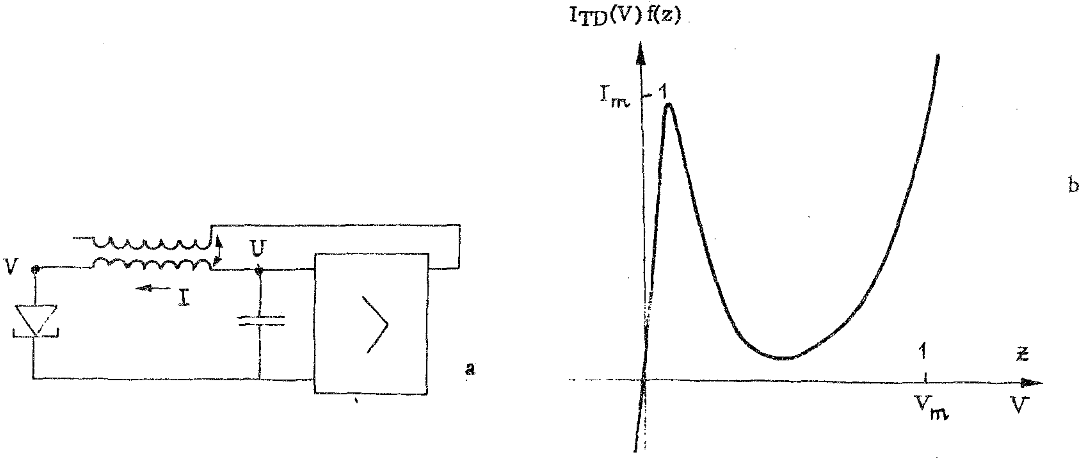

Fig. 1

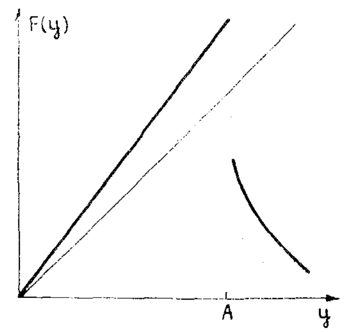

Fig. 2

selves to the graphic mapping of Fig. 2. It is seen that the mapping $y \to F(y)$ consists of two parts. For $y < A$ the mapping is linear: $F(y) = \alpha y$, $\alpha > 1$. Iterations with $y < A$ correspond to an exponential buildup of oscillations in the circuit, in which the current through the diode remains lower than the threshold I_m. For $y > A$, $F(y)$ is a nonlinear function describing the discharge of oscillatory energy in the diode. The stretching nature of the mapping ($|F'(y)| > 1$) assures a random regime of oscillations. The process consists of a sequence of trains of rising oscillations, with the number of oscillations in each train differing and being a random quantity.

In the presence of noise, the procedure of the transition to mapping the sequence is not rigorously valid mathematically. With low noise, however, the process differs little from a purely dynamic one. To describe it, therefore, we can, as before, use the mapping of Fig. 2, adding the noise [6]. As a result, we obtain a discrete one-dimensional mapping with noise,

$$y_{n+1} = F(y_n) + g\xi_n, \tag{2}$$

where g^2 is proportional to the noise intensity and ξ_n is a sequence of independent random quantities, $\langle\xi\rangle = 0$ and $\langle\xi^2\rangle = 1$. Equation (2) is the fundamental equation for describing the statistical characteristics of the self-oscillations of the generator in the presence of noise.

3. In deriving the statistical characteristics of the system (2), we shall start from the general relations for discrete mappings with noise, established in [7]. We designate the distribution function of the random quantity ξ as $V(\xi)$. Then the equation for the evolution of the probability density $P(y, n)$ of the quantity y has the form

$$P(y, n+1) = \int duK(y, u)\, P(u, n), \tag{3}$$

where $K(y, u) = g^{-1}V\{g^{-1}[y - F(u)]\}$. In the limit $n \to \infty$, any sufficiently smooth initial distribution tends toward an invariant probability density $P_0(y)$. Knowledge of $P_0(y)$ is not enough to find certain statistical characteristics, however, For example, an important characteristic, and easily observable experimentally, is the distribution W(n) of the number of oscillations in a train [3, 8]. In a determinate system ($g \equiv 0$), the number of oscillations in a train, larger by one than the number of iterations of the mapping with $y < A$, is a one-to-one function of the starting point of the train (i.e., the point y_n for which $y_{n-1} > A$). Therefore, to determine W(n) it is enough to know $P_0(y)$. For $g \neq 0$, however, the situation changes. Here the statistical problem of escape from a region arises. In fact, let the train start from a certain point $y < A$. Then the number of oscilaltions in the train is one larger than the number of iterations of the mapping (2) in the region $y < A$, and because of the noise it is not a determinate function of y. It makes sense to talk about the probability Q(y, n) that this number equals n. Obviously, the normalization condition

$$\sum_{n=1}^{\infty} Q(y, n) = 1 \tag{4}$$

must be satisfied. The normalized distribution $P_1(y)$ of initial amplitudes in the train is determined from the invariant distribution function:

$$P_1(y) = \frac{\bar{P}_1(y)}{\int \bar{P}_1(y)\,dy}, \quad \bar{P}_1(y) = \int_{u>A} du K(y, u) P_0(u). \tag{5}$$

Knowing $P_1(y)$ and Q(y, n), one can find the unknown distribution function of the number of oscillations in a train:

$$W(n) = \int P_1(y) Q(y, n-1) dy. \tag{6}$$

The nontrivial problem consists in the calculation of Q(y, n). To find this function, we use an equation for the transition probability derived in [7]. Let p(u, y; k) be the probability that the system makes the transition from point y to point u in k steps, always remaining in a given region D of phase space in the process (in our case, D is the region of $y < A$). This quantity satisfies the equation [7]

$$p(u, y; k) = \int_D dw K(w, y) p(u, w; k-1). \tag{7}$$

Then the probability that the system leaves the region D in exactly n iterations is

$$Q(y, n) = \int_D du [p(u, y; n) - p(u, y; n+1)]. \tag{8}$$

Substituting (7) into (8), after simple transformations we obtain for Q(y, n) the recurrent equation

$$Q(y, n) = \int_D du K(u, y) Q(u, n-1) \tag{9}$$

with the initial condition

$$Q(y, 1) = 1 - \int_D K(u, y) du. \tag{10}$$

We note that the arguments in the kernel K in (9) are rearranged in comparison with those of (3). The equations obtained solve the stated problem. The functions Q(y, n) are found from (9) and (10), the initial distribution $P_1(y)$ is determined from (5), and the invariant distribution $P_0(y)$ is found from (3).

4. Let us consider the case of low noise, using the perturbation method. Let the noise distribution function $V(\xi)$ be a symmetric function that decreases sufficiently rapidly. Then the so-called differential approximation as $g \to 0$ can be used:

$$\frac{1}{g} V\left(\frac{y}{g}\right) = \delta(y) + \frac{1}{2} g^2 \delta''(y), \tag{11}$$

TABLE 1

n	$W(n)$			
	$g=0$	$g=0{,}005$	$g=0{,}01$	$g=0{,}02$
2	0	0,03	0,03	0,09
3	0,524	0,5	0,47	0,47
4	0,476	0,44	0,44	0,34
5	0	0,03	0,06	0,09
6	0	0	0,0001	0,01

$$K(y,u) = \delta(y - F(u)) + \frac{1}{2} g^2 \delta''(y - F(u)). \tag{11}$$

Let $P_0(y) = P_0^0(y) + g^2 P_0^1(y)$, where $P_0^0(y)$ is the invariant probability density in the absence of noise. Substituting this expression together with (11) into (3), for the correction $P_0^1(y)$ we obtain

$$P_0^1(y) = \frac{1}{2} \frac{P_0^{0''}(u)}{|F'(u)|}, \quad u = F^{-1}(y). \tag{12}$$

We obtain a similar expression for the correction to the distribution function $P_1(y) = \bar{P}_1^0(y) + g^2 \bar{P}_1^1(y)$:

$$\bar{P}_1^1(y) = \frac{1}{2} \frac{P_0^{0''}(u)}{|F'(u)|}, \quad u = F^{-1}(y), \quad u > A. \tag{13}$$

To determine the functions $Q(y, n)$, we use the linearity of the mapping in the region D: $F(y) = a y$. Then, with allowance for (11), (9) takes the form

$$Q(y,n) = Q(ay, n-1) + \frac{1}{2} g^2 Q''(ay, n-1) \tag{14}$$

with the initial condition

$$Q(y,1) = \theta(ay - A) + \frac{1}{2} g^2 \delta'(ay - A),$$

where θ is the Heaviside function.

Iterating (14), we obtain the expression for $Q(y, n)$ in explicit form:

$$Q(y,n) = Q^0(y,n) + g^2 Q^1(y,n) = \theta(A - a^{n-1}y)\theta(a^n y - A) + \\ + \frac{1}{2} g^2 \left[\frac{1 - a^{2n}}{1 - a^2} \delta'(a^n y - A) + \frac{1 - a^{2n-2}}{1 - a^2} \delta'(A - a^{n-1} y) \right]. \tag{15}$$

As a result, substituting (15) into (6), we obtain

$$W(n) = W^0(n) + g^2 W^1(n), \tag{16}$$

where

$$W^0(n) = \int P_1^0(y) Q^0(y, n-1)\, dy,$$

$$W^1(n) = \int (P_1^1(y) Q^0(y, n-1) + P_1^0(y) Q^1(y, n-1))\, dy.$$

From (16) it is seen that the two factors – the change in the invariant distribution function and the change in the time in which the boundary of the region is reached – make contributions of the same order to the correction to the distribution function of the numbers of oscillations in a train.

We draw attention to the fact that the derivatives of the invariant distribution function $P_0^0(y)$ appear in the result. It may happen, however, that this function is not differentiable at certain points (see [4]). Then the differential approximation (11) is not applicable, strictly speaking. In this case, however, one can replace quantities of the type $\delta'(\xi)$ by functions $d(g^{-1}V(\xi/g))/d\xi$ in the answer (16). Through integration, it is found that the contribution from discontinuities of the function $P_0^0(y)$ of the first kind will be of the order of g, rather than g^2 as in the case of points of continuity.

5. In conclusion, we give the results of a calculation of W(n) for a concrete mapping. The function F(y) was chosen in the form $F(y) = \alpha y$ for $y < 1$ and $F(y) = [\alpha^2 + \alpha - 1 - y(\alpha + 1)]\alpha^{-3}$ for $y > 1$. With such a choice of the mapping parameters, the invariant distribution function can be written analytically [4, 5]; there can be three or four oscillations in a train, with $W^0(3) = \alpha/(1 + \alpha)$ and $W^0(4) = 1/(1 + \alpha)$. A numerical calculation was made from Eqs. (3), (5), (9), and (10) for the value $\alpha = 1.1$ and for different intensities g of the Gaussian noise. From the values of W(n) presented in Table 1, it is seen how the distribution function W(n) "broadens" with an increase in g (this phenomenon was observed experimentally in [8]).

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PARAMETRIC INTERACTION OF WAVES IN A PLANE PLASMA WAVEGUIDE WITH RANDOM DENSITY INHOMOGENEITIES

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The interaction of electromagnetic waves in a plane plasma waveguide with one-dimensional random inhomogeneities is investigated. Equations for the mean field amplitudes over inhomogeneity ensemble are derived. The threshold of the decay instability is determined. Some estimates are presented for laboratory plasma.

The interaction of quasimonochromatic waves and the propagation of nonlinear signals in an unbounded plasma with random inhomogeneities have been investigated very thoroughly (for example, see [1]). In real conditions the plasma is bounded; therefore, it is of interest to analyze the effect of bounded nature of the plasma system on the parametric interaction of waves taking into consideration the random inhomogeneities in the plasma. It is known (see [2, 4, 5]) that the boundedness of the plasma (plasma waveguides) leads to qualitatively new physical effects manifested in additional dispersion of the system modes and also to polarization selection rules for the three-dimensional resonance mode interaction processes in a plasma waveguide. In the present article the rigorous nonlinear boundary-value problem of wave interaction in a plane plasma waveguide is solved for the case when the waveguide is filled with "cold" plasma having random one-dimensional density inhomogeneities. A systematic derivation of the equations of the mean field amplitude is given and the threshold of the decay instability is determined for the electromagnetic waves in the waveguide. Estimates

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