EVOLUTION OF THE POWER SPECTRUM IN THE PERIOD-DOUBLING ROUTE TO CHAOS

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UDC 517.938;537.86:519

We discuss the development of the power spectrum in the period-doubling route to chaos for systems with different time-shift symmetry properties. We introduce a universal law for the evolution of discrete spectral components prior to the transition in systems with continuous symmetry. It is shown that in the absence of continuous symmetry, discrete spectral lines assume a Lorentzian shape for certain parameter values, and they increase in width with a critical exponent of 1/2. We discuss the relation between a central peak in the spectrum and the breakdown of symmetry in a chaotic phase space. The results derived enable us to sketch a complete picture of the evolution of the spectrum.

Introduction. The transition to stochastic behavior has recently been of considerable interest [1, 2]. One of the most frequently encountered scenarios for such a transition is a series of period-doubling bifurcations leading to chaos. This sort of transition has been observed in many experiments [3-8]. According to the theory put forth by Feigenbaum [9], period-doubling bifurcations obey a set of quantitative universal laws. Within the scope of this theory, a number of universal constants have been found which describe the evolution of the observed quantities. The calculation of the power spectrum is of special interest, as it can conveniently be measured experimentally. Some of the laws governing the evolution of the power spectrum were found in [10-17].

In the present paper, we discuss a number of spectral characteristics which are intimately related to the symmetry properties of a dynamic system. A major role is played by the difference between two types of time-shift symmetry, continuous and discrete. Autonomous continuous-time systems described by ordinary differential equations will be called systems with continuous symmetry. These are invariant under time shifts of any magnitude. Time-periodic nonautonomous systems and discrete mappings will be called systems with discrete symmetry. These are invariant only under time shifts which are a multiple of the period (or a multiple of unity for mappings).

In some respects, the difference between these two types of symmetry is unimportant. For example, period-doubling bifurcations in either case obey the same universal law,

$$r_c - r_n \sim \delta^{-n}, \quad \delta = 4{,}669\ldots, \tag{1}$$

where the r_n are the bifurcation values, and r_c the critical value of the parameter. This universality results from the possibility of reduction to a one-dimensional mapping. For systems with continuous symmetry, the reduction takes place through iterative mapping: we choose a secant in phase space, and mark points only where the trajectory intersects it. In the discrete-symmetry case, the mapping is obtained by the so-called stroboscopic method, where the values of the variables are marked every period. We stress that the inverse transformation from a mapping to the original system is not unique, since that requires knowing the time along the trajectory. Here the difference between the two types of symmetry is manifested directly. For the discrete symmetry case, the time between two successive points of a mapping is fixed (and equal to the external forcing period). Physically, this means that period-doubling occurs only in amplitude modulation of a process — there is no phase modulation. For continuous symmetry, the rate of motion along a trajectory is in general not constant, because of a lack of commensurability between oscillations. This means physically that there can be phase modulation as well as amplitude modulation.

Institute of Applied Physics, Academy of Sciences of the USSR. Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 29, No. 12, pp. 1438-1446, December, 1986. Original article submitted July 4, 1985.

 0033-8443/86/2912-1076$12.50

The clearest difference of all between the two symmetry types shows up in the power spectrum. In particular, for chaotic amplitude modulation, a wideband component appears in the spectrum, but discrete lines are retained. On the other hand, for chaotic phase modulation, the spectrum is purely continuous. Previous work has considered the behavior of the spectrum prior to the transition to chaos in systems with discrete symmetry [10-15], as well as line broadening following the transition in the continuous-symmetry case [16, 17]. In order to obtain a complete picture, it is necessary to consider the development of the spectrum prior to the transition in systems with continuous symmetry, and to describe line broadening in the discrete-symmetry case. These problems are settled in the present paper. In section 1, a universal law is derived for the evolution of discrete components of the power spectrum prior to the transition to chaos in systems with phase modulation. In Sec. 2, we discuss the transition from a discrete to a continuous spectrum for systems with discrete symmetry. We demonstrate that for certain parameter values, the discrete lines become Lorentzian in shape, and they increase in width with a critical exponent of 1/2. We demonstrate in Sec. 3 that when the symmetry of the chaos is destroyed, a peak appears in the power spectrum at zero frequency.

1. Spectrum Prior to the Transition to Chaos. Prior to the transition to chaos, the spectrum consists of discrete components, which increase in number with every doubling bifurcation. A universal law for this process was found in [10, 11] for the case of pure amplitude modulation (i.e., for systems with discrete symmetry): the total intensity S_n of spectral components appearing at the n-th bifurcation is governed near the critical point by the similarity law

$$S_n \sim \beta^{-n}, \quad \beta = 10{,}48... \tag{2}$$

The constant β is approximately given in terms of Feigenbaum's universal constant α = 2.5029... by

$$\beta \simeq 2\alpha^4/(\alpha^2+1) = 10{,}8...$$

In the present section, we generalize (2) to the case of phase modulation.

As noted above, phase modulation occurs in systems with continuous symmetry because of incommensurability of the periods involved. From the standpoint of the transition to a one-dimensional iterative mapping, this means that the time along the trajectory from one crossing of the secant to the next depends on the intersection point. In general, amplitude and phase modulation can be represented in the form

$$\bar{f}(t) = A(t)\, g(\varphi(t)),$$

where $g(\tau) = g(\tau + 2\pi)$ is a 2π-periodic function which describes the shape of the variation, $A(t)$ is the amplitude, and $\varphi(t)$ is the phase, the rate of change of which is given by the instantaneous frequency

$$\dot{\varphi} = \omega(t). \tag{3}$$

We will assume that both the amplitude and phase modulation are small, i.e., $A(t) = 1 + \varepsilon A^1(t)$, $\omega(t) = 1 + \varepsilon\omega^i(t)$, $\varepsilon \ll 1$ (note that while this may not be true of the original process, it certainly holds after the first few doublings, since the degree of modulation is reduced by successive doublings). Then to first order in ε,

$$\bar{f}(t) = g\left(t + \varepsilon \int^{t} \omega^1(t)\, dt\right) + \varepsilon A^1(t)\, g(t),$$

i.e., amplitude and phase modulation can be treated independently. Since the spectrum for amplitude modulation has already been considered [10, 11], we confine our attention to a process which is purely phase-modulated:

$$f(t) = g(\varphi(t)). \tag{4}$$

We assume that the frequency ω is approximately constant over each loop of the trajectory from one traversal of the secant to the next (this is assured by an appropriate choice of function $g(\tau)$), and is defined by the intersection point. For period doubling, therefore, the frequency obeys the recurrence relation

$$\omega_{i+1} = F(\omega_i) \tag{5}$$

and the quantities ω_i possess the universal properties derived by Feigenbaum [15] for one-dimensional mappings. For period-doubling bifurcations, there is a switch from an N_n-loop cycle to an N_{n+1}-loop cycle, where $N_n = 2^n$. We denote the period of the N_n-loop cycle by T_n. By virtue of the definition introduced for phase,

$$\varphi(T_n)-\varphi(0)=\int_0^{T_n}\omega(t)\,dt=N_n 2\pi. \tag{6}$$

The spectrum of a process with period T_n consists of harmonics at frequencies $2\pi k/T_n$, $k = 0, \pm1, \pm2, \ldots$. In switching to a cycle of period T_{n+1}, the previous set of harmonics is retained and new ones appear at frequencies $2\pi\ell/T_{n+1}$, $\ell = \pm1, \pm3, \pm5, \ldots$. Thus, the new harmonics are odd-index. Any T-periodic function $f(t)$ can be decomposed into two parts:

$$f(t)=f_1(t)+f_2(t),\quad f_1(t)=(1/2)\,[f(t)+f(t+T/2)],$$
$$f_2(t)=(1/2)\,[f(t)-f(t+T/2)],$$

where f_1 consists solely of even harmonics and f_2 of odd. The total power S in the odd harmonics is

$$S=(1/T)\int_0^T f_2^2(t)\,dt. \tag{7}$$

Our problem is to calculate the way in which S decreases with successive period doublings. First of all, we transform S to a more convenient form. We introduce the notation

$$\varkappa(t)=\varphi(t+T/2)-\varphi(t)-\pi N, \tag{8}$$

since near a critical point a newly emergent cycle differs but little from its predecessor: $|\varkappa(t)| \ll 1$. Then to a first approximation we obtain

$$S\simeq 1/4T\int_0^T [g'(\varphi(t))]^2\varkappa^2(t)\,dt\simeq h/T\int_0^T \varkappa^2(t)\,dt, \tag{9}$$

where

$$h=(1/8\pi)\int_0^{2\pi}[g'(\tau)]^2 d\tau.$$

Substituting (8) into (9) and making use of (3), we obtain

$$S=h/T\int_0^T c^2(t)\,dt, \tag{10}$$

where

$$c(t)=\int_t^{t+T/2} d(\tau)\,d\tau,\quad d(t)=\omega(t)-\omega(t+T/2).$$

Note that this function $d(t)$ is just that part of the function $\omega(t)$ which corresponds to the newly emergent odd harmonics.

We now exploit the fact that the process $\omega(t)$ obeys the universal laws. Specifically, Feigenbaum [15] shows that the quantities $d_n(t)$ and $d_{n+1}(t)$, belonging to the N_n- and N_{n+1}-cycles respectively, satisfy the relation

$$d_{n+1}(t)=d_n(t)\,\sigma(t/T_{n+1}), \tag{11}$$

where $\sigma(x)$ is a universal function. To good accuracy, σ can be approximated by a piecewise-linear function [10, 15]:

$$\sigma(x)\simeq\begin{cases}\alpha^{-2}, & 0\leqslant x<1/4\\ \alpha^{-1}, & 1/4\leqslant x<1/2\end{cases},\quad \sigma(x+1/2)=-\sigma(x). \tag{12}$$

Making use of (11) and (12), it is straightforward to derive the relation between the quantities $c_{n+1}(t)$ and $c_n(t)$ (the subscript n refers to the N_n-cycle):

$$c_{n+1}(t)=\begin{cases}-(1/\alpha)\,c_n(0)+(1/\alpha^2)\,c_n(t), & 0<t<T_{n+1}/4\\ -(1/\alpha^2)\,c_n(0)+(1/\alpha)\,c_n(t), & T_{n+1}/4<t<T_{n+1}/2\end{cases},$$
$$c_{n+1}(t+(T_{n+1}/2))=-c_{n+1}(t). \tag{13}$$

Substituting (13) into (10), we immediately obtain

$$S_{n+1}=((\alpha^2+1)/2\alpha^4)\,(S_n+hV_n),\quad V_{n+1}=(\alpha-1)^2\alpha^{-4}V_n, \tag{14}$$

where $V_n = c_n^2(0)$. Equations (14) are indeed the desired generalization of the spectral similarity law to the case of systems with phase modulation. Substituting a solution of the form S_n, $V_n \sim q^n$ into the linear mapping (14), we find the eigenvalues

$$q_1^{-1}=\beta=\frac{2\alpha^4}{\alpha^2+1}=10{,}8\ldots,\quad q_2^{-1}=\gamma=\left(\frac{\alpha^2}{\alpha-1}\right)^2. \tag{15}$$

Thus, the general solution of (14), a superposition of the two linearly independent solutions with constants q_1 and q_2, may be expressed in terms of two universal constants, in contrast to (2):

$$S_n\sim S_{01}\beta^{-n}+S_{02}\gamma^{-n}. \tag{16}$$

We have calculated an accurate value for the constant γ, equal to 21.02..., using the original relations (11); since $\gamma > \beta$, for large n the contribution of the second term in (16) can be neglected, and (16) reduces to (2). Thus, the fall-off behavior of the power in the newly engendered spectral components near a critical point is independent of the type of symmetry possessed by the system. We stress, however, that the relation derived by Feigenbaum [10] between isolated spectral components in only valid for systems having discrete symmetry.

The universal constant γ was first obtained in [17], which treated spectral line broadening beyond the transition to chaos for the phase-modulation case. There it was shown that for parameters $\bar{r}_n$ which are "mirror images" of the r_n, satisfying the Feigenbaum relation $\bar{r}_n - r_c \sim \delta^{-n}$, the linewidth Δ_n varies in a universal manner:

$$\Delta_n\sim(2\gamma)^{-n}. \tag{17}$$

This law can be expressed in terms of a critical exponent:

$$\Delta\sim(r-r_c)^{\rho},\quad \rho=(\log(2\gamma)/\log\delta)=2{,}42\ldots \tag{18}$$

We present here a simple derivation of Eqs. (17) and (18). We see from (3) that the phase is an integral of the frequency ω. When ω varies chaotically, therefore, the rate of diffusive spreading of ϕ equals the value of the continuous spectrum of the process $\omega(t)$ at zero frequency. A derivation was derived in [14] for the similarity law governing the continuous part of the spectrum $s(\nu)$,

$$s_{n+1}(\nu)=\frac{1}{2}\left|\frac{1-\exp(i\nu)\alpha^{-1}}{\alpha^2}\right|^2 s_n(\nu).$$

For the zero-frequency component, we then have $s_{n+1}(0) = (2\gamma)^{-1}s_n(0)$. In turn, the line width is inversely proportional to the rate of diffusive spreading of the phase, whence we immediately obtain Eq. (17).

2. Transition to a Continuous Spectrum in Systems with Discrete Symmetry. For discrete symmetry, even in the chaotic regime, the power spectrum has a delta-function component besides the continuous component. Qualitatively, the transition from a discrete to a continuous spectrum takes place as follows. At each doubling, with parameter values r_1, r_2, ..., r_n, ..., new discrete peaks appear, so that at the critical point the spectrum is discrete but infinitely dense. The discrete spectral components are transformed into a continuous spectrum at the "mirror image" parameter values $\ldots\bar{r}_n\ldots$, $\bar{r}_2$, $\bar{r}_1$ in the reverse order of their appearance. We describe only the transformation at $r = \bar{r}_1$; the remainder are similar. (These transformations are sometimes known as "band merging.")

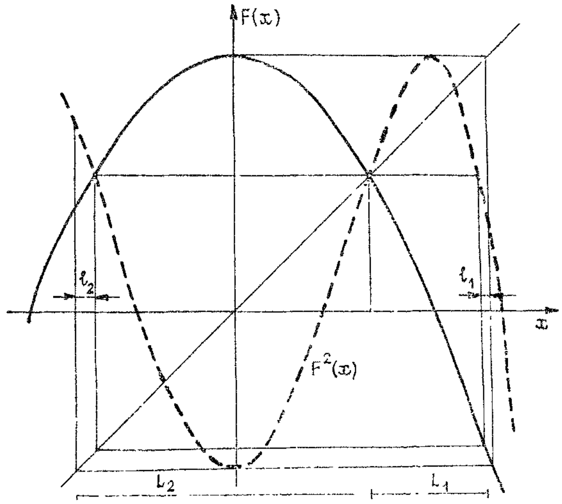

Fig. 1. Structure of the mapping near a point of disappearance of a discrete spectral peak.

For the sake of definiteness, we consider the discrete mapping $x_{i+1} = F(x_i)$. For $r < \bar{r}_1$, the attractor consists of the two intervals L_1 and L_2, with $F^2(L_1) = L_1$, $F^2(L_2) = L_2$. Accordingly, there is a discrete peak in the spectrum at a frequency $\omega = \pi$. When $r = \bar{r}_1$, the intervals L_1 and L_2 coalesce, and for $r > \bar{r}_1$ there is a single invariant interval $L = L_1 + L_2$. When r is only slightly greater than $\bar{r}_1$, the point x leaves $L_1(L_2)$ under the influence of F^2 only if it lies within the small interval ℓ_1 (ℓ_2) (see Fig. 1). Thus, metastable chaotic sets arise to take the place of the invariant segments L_1 and L_2 [18]. The transition from a strange attractor to a metastable chaotic set is known as a crisis in two symmetric attractors. A number of statistical characteristics of metastable chaotic sets have been considered in [18-20], but to the best of our knowledge, the power spectrum has heretofore not been calculated.

The lifetime in a metastable chaotic set is governed by a Poisson distribution [18]. The probability p of leaving the interval L_1 under the influence of F^2 is proportional to the probability of lying in the interval ℓ_1:

$$p = (\mu(l_1)/\mu(L_1)) \sim (r - \bar{r}_1)^{1/2}, \tag{19}$$

where μ is an invariant probability measure. By symmetry, the probability of leaving the interval L_2 is also p, since $\mu(\ell_2) = \mu(\ell_1)$, $\mu(L_2) = \mu(L_1)$. The Poisson process is completely defined by the quantity p, and for large k,

$$\mathrm{prob}\,(x \in L_1, F^k(x) \in L_1) = \mathrm{prob}\,(x \in L_2, F^k(x) \in L_2) = (1/2)\,(1-(-1)^k \exp(-pk)). \tag{20}$$

Based on (20), we can calculate the shape of the spectral peak at the frequency $\omega = \pi$. A narrow spectral peak is determined by the asymptotic behavior of the autocorrelation function $R(j) = \langle x_i x_{i+j} \rangle - \langle x \rangle^2$ at large j. We represent x_i in the form (as in [20])

$$x_i = u_i + y_i,$$

where

$$u_i = \begin{cases} U_1 = \int_{L_1} x\,d\mu, & \text{for} \quad x_i \in L_1 \\ U_2 = \int_{L_2} x\,d\mu, & \text{for} \quad x_i \in L_2 \end{cases}. \tag{21}$$

The function u_i describes slow transitions from one metastable chaotic set to another, while y_i describes the detailed chaotic motion. In calculating the correlation function for large j, we must take only the correlation properties of u_i into account. Making use of (20), we find

$$R(j) = \langle u_i\, u_{i+j} \rangle - (U_1 + U_2)^2 = (-1)^j \exp(-pj)\,(U_1 - U_2)^2. \tag{22}$$

A Fourier transform gives the spectrum

$$S(\omega)=2\sum_{j=0}^{\infty}R(j)\cos\omega j=(U_1-U_2)^2\left[1+\frac{\operatorname{sh}p/2}{\operatorname{ch}p/2-\cos(\omega-\pi)}\right]. \tag{23}$$

For $p \ll 1$ and $\omega \approx \pi$, this simplifes to

$$S(\omega)\simeq(U_1-U_2)^2\frac{p}{p^2+(\omega-\pi)^2}. \tag{24}$$

Thus, when $r \gtrsim \bar{r}_1$, the discrete peak is transformed into a Lorentzian line. The linewidth increases according to

$$\Delta\omega\simeq p\sim(r-\bar{r}_1)^{1/2}, \tag{25}$$

while the height remains constant. All of the other discrete peaks are broadened in an entirely analogous manner.

3. Emergence of a Central Peak as a Result of Symmetry Breaking of Chaos. The transition from a discrete to a continuous spectrum considered in section 2 can be interpreted as the breaking of discrete time-shift symmetry (see below). Here we show that a similar phenomenon in systems with phase-space symmetry leads to the appearance of a central peak. Many dynamic systems possess one phase-space symmetry group or another. One example of this would be the Lorentz system [21]. The simplest system with phase-space symmetry is the periodically forced nonlinear oscillator

$$\ddot{x}+\gamma\dot{x}+(dU(x)/dx)=\varepsilon\cos\omega t \tag{26}$$

with a symmetric potential, $U(x) = U(-x)$. Such systems can support a strange attractor which is invariant to symmetry transformations, or a set of symmetric strange attractors. The transformation from one type of behavior to another is naturally interpreted as symmetry breaking [22].

We can grasp the simplest symmetry-breaking mechanism for chaos by looking at the system (26) with the potential $U(x) = -x^2/2 + x^4/4$. This case has been previously investigated both theoretically and experimentally [23, 24]. Periodic oscillations take place in both potential wells for small ε, and these become chaotic when ε increases. At the critical value ε_c, the amplitude of the chaotic oscillations increases by so much that transitions are possible from one well to the other (Fig. 2). Thus, ε_c is a critical point for symmetry breaking of chaos.

In exactly the same way as for the case considered in section 2, the oscillations in each potential well become metastable, and the transition from one well to the other is described by a Poisson probability distribution. By analogy to (21), the oscillations for $\varepsilon \gtrsim \varepsilon_c$ can be represented in the form

$$x(t)=z(t)\,x_+(t), \tag{27}$$

where $z(t) = 1$ when the particle is in the right-hand well, and $z(t) = -1$ when it is in the left; $x_+(t)$ describes the detailed chaotic oscillations. Since the correlation time of $x_+(t)$ is small compared with the characteristic time scale for variations of $z(t)$, the functions $x_+(t)$ and $z(t)$ can be considered to be independent. For the correlation function, we then obtain

$$R(\tau)=\langle x(t)\,x(t+\tau)\rangle=2P(\tau)\eta^2+r(\tau), \tag{28}$$

where $\eta=\langle x_+(t)\rangle$, $r(\tau)=\langle x_+(t)\,x_+(t+\tau)\rangle-\eta^2$ is the correlation function of the fine-scale oscillations, and $P(\tau) = \langle z(t)z(t+\tau)\rangle$. Since $z(t)$ is a random telegraphic Poisson process with time constant τ_0, $P(\tau) = \exp(-\tau/\tau_0)$, whence we have $R(\tau) = \eta^2\exp(-\tau/\tau_0) + \gamma(\tau)$. As a result, we obtain the spectrum

$$S(\omega)=2\int_0^{\infty}R(\tau)\cos\omega\tau d\tau=s(\omega)+\frac{\eta^2\tau_0}{1+\tau_0^2\omega^2}, \tag{29}$$

where $s(\omega)$ is the spectrum of the fine-scale metastable oscillations. Thus, a Lorentzian peak appears at zero frequency when there is symmetry breaking of chaos. The width of the peak ($\Delta\omega \sim \tau_0^{-1}$) is proportional to the probability of leaving the metastable set. As

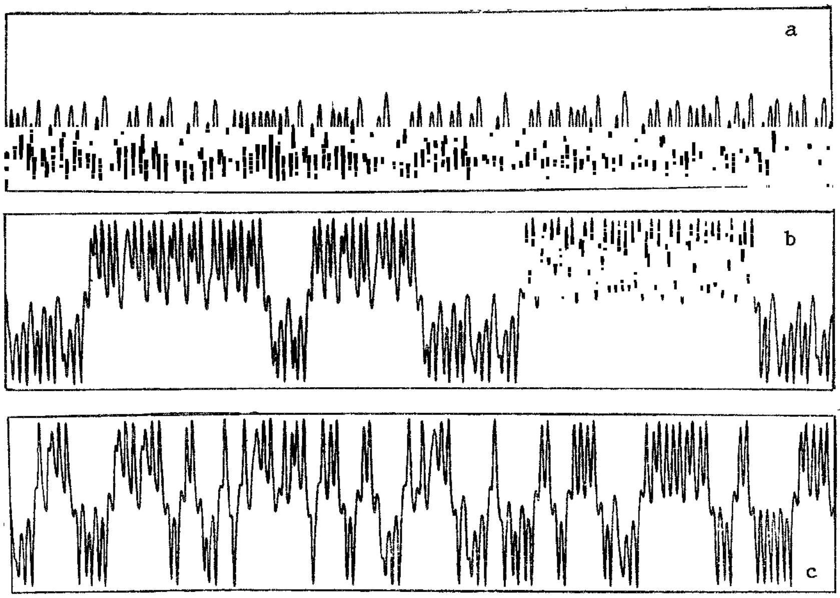

Fig. 2. Stochastic oscillations in the system (26) with $\gamma = 0.5$, $\omega = 1$, and a) $\varepsilon = 0.38$; b) $\varepsilon = 0.386$; c) $\varepsilon = 0.392$. The critical value is $\varepsilon_c = 0.383$.

$\varepsilon \to \varepsilon_c$, we usually have $\tau_0 \sim (\varepsilon - \varepsilon_c)^{-\nu}$, i.e., the peak becomes a delta function. The exponent ν is 1/2 for the system of (26), but this is not universal, and it can take on other values (in the Lorentz system, for example). Similar chaos symmetry-breaking also takes place for symmetry mappings of the type $x_{i+1} = ax_i - x_i^3$.

In this paper, we have shown that the properties of the power spectrum during period doubling are determined by the type of time-shift symmetry a system has. The general picture is as follows.

a) Discrete Symmetry (Nonautonomous Systems with Periodic Time-Dependence and Discrete Mappings). For successive period doubling, new discrete peaks appear, and their intensity is governed by (2). Beyond the critical point of the transition to chaos, the spectrum consists of discrete peaks and a continuous component. The discrete lines start to broaden, according to (25), at quite definite bifurcation points ("band merging" points). Since the line heights remain constant during this broadening, the total power in the continuous spectrum also follows the law (2). Note that the entire transition to chaos can be described as a sequence of symmetry breakings. In fact, the transition from a stationary point to a 2-cycle is a breaking of time-shift symmetry (analogous to the phase transition in an antiferromagnet). The transition from a 2-cycle to a 4-cycle is also a symmetry breaking, and so forth. The quasiperiodic regime which exists at the critical point can thus be interpreted as a state of maximum symmetry breaking (it is not invariant under any temporal displacement). Beyond the critical point, there is a "mirror" sequence of symmetry "reconstructions" — transitions from a 2^{n+1}-part strange attractor to a 2^n-part strange attractor. Under the analogous "reconstruction" of symmetry in phase space, a Lorentization central peak emerges in the spectrum (29).

b) Continuous Symmetry (Autonomous Continuous-Time Systems). Prior to the transition point, the total power in the discrete components which emerge follows the law (16). Broadening starts precisely at the critical point, and is govened by the universal law (18). However, the lines remain rather narrow right up to the more rapid broadening which takes place with "band merging."

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DEPOLARIZATION OF RADAR SIGNAL BACKSCATTERED FROM RANDOM SURFACE

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UDC 537.874.2

The diffraction of a plane electromagnetic wave at an ideally conducting surface with roughness large compared to the wavelength is investigated. The current at each point of the surface is determined in the form of expansion in the small parameter inversely proportional to the wave number, the characteristic radius of curvature of the surface at that point, and the cube of the cosine of the local angle of incidence. The scattering cross section for both polarized and depolarized components is obtained for the rough surface.

The tangential plane method ("Kirchhoff's method") [1, 2] is widely used in the investigation of scattering of electromagnetic waves at statistically rough surfaces with roughness large compared to the wavelength λ. In the radar case, when the tangential planes are perpendicular to the wave vector $k = \alpha k$, where $k = 2\pi/\lambda$, the reflected field has the same polarization as the incident field, although experimental data show the presence of a depolarized component in the scattered field [3].

In the present article the current $j(r)$ ($r \in S$) at surface S is not specified in geometrical optics approximation $\mathbf{j}(\mathbf{r}) = 2\mathbf{j}_0(\mathbf{r})$, $\mathbf{j}_0(\mathbf{r}) = (c/4\pi)\,[\mathbf{n}(\mathbf{r})\mathbf{H}_0(\mathbf{r})]$ (c is the speed of light, n is the normal to the surface, H_0 is the magnetic component of the incident field) but is determined from the solution of the integral equation. The diffraction corrections to the polarized component of the reflected signal are determined from the current, and the depolarization component is computed.

Asymptotic Expansion of the Current. We consider the diffraction of a plane electromagnetic wave, whose magnetic component is $\mathbf{H}_0(\mathbf{R}) = \mathbf{h}_0\exp(ik\alpha\mathbf{R})$ ($\mathbf{h}_0$ is the unit polarization vector; here and below the time dependence of the field is omitted), at a sufficiently smooth, i.e. without ridges, breaks, and singular points, ideally conducting surface S. Surface electric currents are induced on S under the action of the primary field, whose

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Translated from Izvestiya Vysshikh Uchebnykh Zavedenii, Radiofizika, Vol. 29, No. 12, pp. 1447-1453, December, 1986. Original article submitted March 3, 1985.

0033-8443/86/2912-1083$12.50