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UNIVERSALITY AND SCALING OF PERIOD-DOUBLING BIFURCATIONS IN A DISSIPATIVE DISTRIBUTED MEDIUM

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A homogeneous medium, consisting of nonlinear elements, demonstrating transition to chaos via period-doubling bifurcations, is considered. The coupling between the elements is supposed to be of a dissipative type, i.e. it tends to equalize their instantaneous states. Using the renormalization group approach, the following scaling law for weakly inhomogeneous states near the critical point is obtained: at each period doubling the spatial scale increases by $\beta = \sqrt{2}$. On the basis of this law the scaling hypotheses for the transition to chaos in the semi-infinite and finite systems are proposed. The scaling properties are verified by the numerical calculations with a simple model.

1. Introduction

Recently there has been a great interest in the study of the transition to chaos in nonlinear systems [1]. One of the most commonly discussed routes to chaos is associated with the infinite sequence of period-doubling bifurcations. Feigenbaum's discovery [2] of the universality in the period doublings stimulated a great deal of theoretical and experimental studies. This route to chaos was observed in many experiments carried out in hydrodynamics [3], acoustics [4], optics [5], electronics [6, 7], chemical kinetics [8], etc. In the present paper we consider transition to chaos in a distributed medium, which may be considered as a discrete or a continuous set of coupled elements, each representing a nonlinear dissipative system which may demonstrate Feigenbaum sequence of period-doubling bifurcations. Let us present several concrete examples of such situations.

1) The simplest dynamical system, that demonstrates period-doubling transition to chaos when its parameter is varied, is the one-dimensional mapping

$$u_{n+1} = f(u_n, \lambda), \quad \text{e.g. } u_{n+1} = \lambda(1 - 2u_n^2). \tag{1}$$

Here n is the discrete time, λ is the parameter. The mapping (1) is used, in particular, for the description of population dynamics [9], the variable u_n therewith denotes the deviation of the population level from a certain value. Considering a spatially distributed population in the presence of organism diffusion and the dependence of the population level on a spatial coordinate, we obtain just a system of the described type.

2) The kinetics of chemical reactions is governed by nonlinear reaction–diffusion equations such as

$$\frac{\partial v_i(\boldsymbol{r}, t)}{\partial t} = B_i(v_1, \ldots, v_k, t) + D_i \Delta v_i, \quad i = 1, \ldots, k, \tag{2}$$

where the v_i are the concentrations of reacting components, which depend, generally, on a spatial coordinate $\boldsymbol{r}$ and t. For a spatially homogeneous regime eq. (2) reduce to a system of ordinary differential equations, where transition to chaos via period-doubling bifurcations may take place.

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quently, the convergence rate of the phase volume per period is great. At the period-doubling bifurcation of a limit cycle one of its multiplicators crosses the unit circle at the point -1. Since the phase volume convergence rate in the vicinity of the cycle is proportional to the product of multiplicators, all multiplicators but one should be small. The larger the period of the cycle considered the nearer they are to zero. Therefore, after a sufficiently large number of period-doubling bifurcations, the dynamics of the point system will be effectively one-dimensional. It follows from these considerations that period-doublings in the infinite distributed medium should be treated thoroughly. Really, if disturbances with long wavelengths evolve slowly, their dynamics can not be separated from the dynamics on the leading one-dimensional manifold.

The system of uncoupled one-dimensional mappings may be considered as a model of a medium of uncoupled elements. Then we may introduce a coupling. It is convenient to impose the following properties of the coupling:

1) The coupling between the elements is symmetric.

2) The coupling does not influence the dynamics of a spatially homogeneous solution, i.e. when instantaneous states of the elements are equal, the coupling force vanishes.

3) The coupling has a finite spatial range, i.e. it tends to zero rapidly enough with the increase of the distance between the elements.

4) The coupling is dissipative and provides the equalization of instantaneous states of the elements.

Thus, the generalization of the mapping (1) can be used to investigate the transition to chaos in a distributed system with a dissipative coupling of the elements. In fact, the leading one-dimensional manifold determines the dynamics of the transition in the point system. In the case of a distributed medium the dynamics of each element is also associated with this one-dimensional manifold, but the states of the elements are different and this difference gives some corrections to the dynamics. Thus, we come to the following operator equation:

$$u_{n+1}(x) = \hat{F}[u_n(x)] = \hat{m} f(u_n(x), \lambda). \qquad (5)$$

Here x is the spatial coordinate, $f(u, \lambda)$ is the nonlinear function demonstrating transition to chaos via period-doublings in (1), $\hat{m}$ is the linear operator, which describes the coupling.

In the case of discrete variables x the operator $\hat{m}$ can be written in the following way:

$$\hat{m} u(x) = \sum_j c_j u(x-j), \qquad (6)$$

while in the case of continuous x we have

$$\hat{m} u(x) = \int c(y) u(x-y) \, \mathrm{d}y. \qquad (7)$$

The properties of the coupling given above impose the following requirements on the operator:

1) Symmetry: $c_j = c_{-j}$, $c(y) = c(-y)$.

2) Normalization: $\Sigma_j c_j = 1$, $\int c(y) \, \mathrm{d}y = 1$.

3) Locality: the sum $\Sigma_j j^2 c_j = \Delta^2$ or the integral $\int y^2 c(y) \, \mathrm{d}y = \Delta^2$ are finite. The value Δ defines the characteristic space scale associated with the operator $\hat{m}$ – diffusion length per unit discrete time.

4) Dissipativity: the spectrum $m(k) = \mathrm{e}^{-\mathrm{i}kx} \hat{m} \, \mathrm{e}^{\mathrm{i}kx}$ lies in the unit circle: $|m(k)| < 1$.

From 1)–4) we unequivocally obtain the form of the spectrum $m(k)$ in the region small wave numbers

$$m(k) = 1 - \tfrac{1}{2}(\Delta k)^2 + \cdots . \qquad (8)$$

We expect that eq. (5) permits us to describe quantitatively the properties of distributed dissipative systems near the point of transition to chaos via period-doublings, irrespective of a concrete type of dynamic equations of the elements and of the way in which coupling appears (the coupling should only satisfy the conditions given above). This point is confirmed by the renormalization group analysis presented below. The condition of dissipativity 4) is the most nontrivial one. How-

In the general case we have a continuous set of such systems, coupled with diffusion. Note that similar to (2) equations are used for the description of nonequilibrium processes in solid-state physics.

3) Many systems of different nature are described by a differential-delay equation (see examples in [10–12]):

$$\varepsilon \dot{v} + v = f(v(t-T), \lambda). \tag{3}$$

Let n and ξ be an integer part and a fractional one of the quantity $t(T+\varepsilon/2)^{-1}$, respectively. Then, for $\varepsilon \to 0$, we may reduce (3) to a set of uncoupled mappings $v_{n+1}(\xi) = f(v_n(\xi), \lambda)$, where $v_n(\xi) = v(t)$. For each fixed ξ we have a mapping of the type (1), which demonstrates period-doublings when parameter λ changes. The influence of the term $\varepsilon \dot{v}$ in (3) can be considered as the appearance of the coupling between these mappings. The role of the space coordinate in this case is played by ξ.

4) The long Josephson function in the periodic external field is described by a nonautonomous sine-Gordon equation [13].

$$v_{tt} - v_{xx} + \gamma v_t + \sin v = A \cos \omega t. \tag{4}$$

The equation for spatially homogeneous oscillations coincides with the equation for a point Josephson junction; its numerical investigation reveals a transition to chaos via period doublings [14]. The long junction can be considered as the continuum set of interacting point systems. Analogous equations are commonly used for description of the crystal dynamics (considered as a lattice of nonlinear oscillators) in intense acoustic or light field.

The examples presented show that we deal with a large class of distributed systems, worthy of special consideration. We may advance using the universal properties at the period-doublings, which point (uncoupled) systems exhibit. Consequently, we may expect some universality to remain after introducing the coupling. We show in this paper, that the distributed systems with dissipative coupling exhibit universal properties at the period-doubling transition to chaos. By a dissipative coupling we mean such a coupling that tends to equalize instantaneous states of the interacting systems. In the above examples systems 1, 2, 3 are of a dissipative coupling type.

In section 2 the simplest model is presented which reproduces adequately the regularities in question. Taking into account that dissipative coupling provides the stability of a spatially homogeneous regime up to the transition point, we develop in section 3 a renormalization group equation for an operator, which governs the evolution of the nearly homogeneous solutions. The fixed point of the renormalization group is found; it yields a universal factor $\beta = 2^{1/2}$, which determines the scale of spatial transformation when time period is doubled. On the basis of this result we formulate in section 4 the scaling hypotheses for spatial structures, arising after successive period-doublings in semi-infinite and finite systems, as well as for corresponding bifurcation values of the parameter. These hypotheses are confirmed by numerical data, presented in section 5. For simplicity, we consider only one spatial coordinate; however, all the results may be generalized for the cases of two and three dimensions.

2. The basic model

In order to construct the simplest model in the considered class of the distributed dissipative systems, we discuss first a medium of uncoupled elements (cells). According to Feigenbaum's theory, each element irrespective of the dimension of its phase space can be described by a one-dimensional mapping, analogous to (1).

Heuristically, this phenomenon may be explained in the following way. We deal with a dissipative system which has the property of the convergence of the phase volume. A characteristic time scale of motions (e.g., period) tends to infinity near the point of transition to chaos. Conseever, its validity for some concrete systems can be ascertained analytically. First, these are electronic systems with delayed feedback [11, 12], in which the signal passes successively through a nonlinear inertialess element, a delay element and a linear filter. Here eq. (5) appears naturally. In particular, the differential-delay equation (3) may be written in the form (5) if we use variables n, ξ (see the introduction). Operator $\hat{m}$ is here symmetrical only for small k: $m(k) \simeq 1 - \frac{1}{2}(\varepsilon k)^2 + \cdots$. However, this is sufficient for our theory to work. Another example is the system of reaction–diffusion equations (2). Let $v_i = v_i(t)$ be a spatially homogeneous solution of these equations (zero flux boundary conditions are supposed). For disturbances of the type $w_i(\boldsymbol{r}, t) = w_i(\boldsymbol{k}, t)\exp(\mathrm{i}\boldsymbol{kr})$ in the linear approximation we get from (2) the following equation:

$$\frac{\mathrm{d}w_i(\boldsymbol{k}, t)}{\mathrm{d}t} = A_{ij}w_j(\boldsymbol{k}, t) - D_i k^2 w_i(\boldsymbol{k}, t). \qquad (9)$$

In the case of equal diffusion constants $D_1 = D_2 = \cdots = D$ after the substitution of $w_i(\boldsymbol{k}, t) = Z_i(t)\exp(-Dk^2 t)$ we obtain from (9) exactly the equation for the homogeneous disturbances. Consequently, homogeneous disturbances always grow more rapidly (or damp slower) than the corresponding nonhomogeneous ones, i.e. the coupling provides the equalization of instantaneous states of spatially distributed elements and is of the dissipative type [15].

3. Scaling relation and its consequences

Since the dissipative coupling provides damping of highly inhomogeneous disturbances, we consider weakly inhomogeneous states in the system, described by eq. (5). We'll develop a scaling relation for the dynamics of these states at successive period-doubling bifurcations. Following Feigenbaum [2], we seek for a renormalization group transformation for the operator $\hat{F} = \hat{m}f$.

Let us consider a weakly inhomogeneous state

$$u(x) = u_0 + \varepsilon u_1(x), \qquad (10)$$

where $\varepsilon \ll 1$, u_0 does not depend on x. After application of the operator $\hat{F}$ to this state we obtain (with accuracy up to ε)

$$\hat{m}f[u(x)] \simeq f(u_0) + \varepsilon f'(u_0)\hat{m}u_1(x) \simeq f[\hat{m}u(x)]. \qquad (11)$$

It follows from the relation (11) that

1) Application of $\hat{F}$ again leads to a weakly inhomogeneous state.

2) For these states $\hat{m}$ and f operations are commutative. Consequently, a two-fold application of the operator $\hat{F}$ yields an operator of the same type

$$\hat{F}^2 u(x) = \hat{m}f[\hat{m}f(u(x))] = \hat{m}^2 f^2[u(x)]. \qquad (12)$$

Now we consider the properties of the operator $\hat{F}$ which appear after its multiple iterations. First, we introduce the scaling operations for the function $u(x)$, namely, $\hat{S}_1$ and $\hat{S}_2$:

$$\hat{S}_1 u(x) = \alpha^{-1}u(x), \quad \hat{S}_2 u(x) = u(\beta^{-1}x). \qquad (13)$$

Using the operator $\hat{F}$ twice, and changing the scale by $\hat{S} = \hat{S}_1\hat{S}_2$, we obtain the operator $\hat{F}_1$:

$$\hat{F}_1 = \hat{S}^{-1}\hat{F}\hat{F}\hat{S}.$$

Then we do the same with the operator $\hat{F}_1$, etc. In consequence we come to the operator recursion relation

$$\hat{F}_n = \hat{S}^{-1}\hat{F}_{n-1}\hat{F}_{n-1}\hat{S}. \qquad (14)$$

According to (12) we rewrite $\hat{F}_n$ in the form

$$\hat{F}_n = \hat{M}_n g_n,$$

where

$$\hat{M}_n = \hat{S}_2^{-1}M_{n-1}^2\hat{S}_2 = \hat{S}_2^{-n}\hat{m}^N\hat{S}_2^n, \qquad (15)$$

$$g_n = \hat{S}_1^{-1}g_{n-1}^2\hat{S}_1 = \hat{S}_1^{-n}f^N\hat{S}_1^n, \qquad (16)$$

where we denoted $N = 2^n$. According to [2], if the parameter λ in $f(u, \lambda)$ equals its critical value λ_c^0

and $\alpha = -2.5029\ldots$ (the Feigenbaum constant), then the sequence g_n converges to the universal function g at $n \to \infty$.

Consider the limiting behavior of the sequence of the operators $\hat{M}_n$. Writing down (15) in the spectral form, we obtain

$$M_n(k) = [m(k\beta^{-n})]^N. \tag{17}$$

If $\beta > 1$, then for large n the form of $M_n(k)$ depends only on the properties of $m(k)$ in the region of small k, i.e. on the form (8). Thus, taking the logarithm of (17), we obtain

$$\ln M_n(k) = -\tfrac{1}{2}\Delta^2 k^2 2^n \beta^{-2n} + \cdots .$$

It is clear tat the regular limit of M_n at $n \to \infty$ exists only if $\beta = \sqrt{2}$:

$$\lim_{n\to\infty} M_n(k) = \exp(-\tfrac{1}{2}\Delta^2 k^2).$$

Summing up, we conclude that the sequence of the operators $\hat{F}_n$ converges at $n \to \infty$ to the universal operator

$$\hat{G} = \exp\left(\tfrac{1}{2}\Delta^2 \frac{\partial^2}{\partial x^2}\right) g$$

which does not depend on the concrete initial operator $\hat{F} = \hat{m}f$ and is the fixed point of the renormalization group equation

$$\hat{G} = \hat{S}^{-1}\hat{G}\hat{G}\hat{S},$$

with constants $\alpha = -2.5029\ldots$, $\beta = \sqrt{2}$. Thus, at successive period-doubling bifurcations the characteristic spatial scale of weakly inhomogeneous states increases by $\beta = \sqrt{2}$.

In conclusion of this section we discuss the consequence of the obtained scaling relation for transition to chaos in the infinite homogeneous medium.

In the point system period doublings accumulate to the critical value λ_c^0. Then at $\lambda < \lambda_c^0$ the homogeneous state in the distributed system, undergoing the same period-doubling bifurcations, will be stable. At $\lambda > \lambda_c^0$ chaotic in time and space oscillations appear. With the doubling of the time scale, the characteristic spatial scale-correlation length r_c increases by β. Since the doubling of the time scale corresponds to the decrease of $\lambda - \lambda_c^0$ by $\delta = 4.6692\ldots$, we obtain at once the critical index for the correlation length

$$r_c \sim (\lambda - \lambda_c)^{-\nu}, \quad \nu = \frac{\log \beta}{\log \delta} = 0.2249\ldots . \tag{18}$$

This result coincides with the expression for the correlation length, obtained in [15] from the linear stability considerations.

4. Scaling hypotheses for bounded systems

4.1. *Infinite system with local inhomogeneity*

The problem of behavior of the infinite medium with a local inhomogeneity arises, for example, in the following cases:

1) There is a local disturbance of the parameter λ.

2) There is a semi-infinite medium $0 \le x < \infty$ with a boundary condition, such as $u(0) = u_0$.

3) There is a solution with a phase dislocation in the infinite homogeneous medium. By the phase dislocation we mean the following. Suppose that the mapping (1) has a stable period-2 cycle (u_1, u_2). Then it is clear that in the presence of small dissipative coupling the distributed system (5) possesses a stable period-2 solution with different phases at $x \to \pm\infty$: $u_{2i}(\infty) = u_{2i+1}(-\infty) = u_1$, $u_{2i}(-\infty) = u_{2i+1}(\infty) = u_2$. Although such a solution with the phase dislocation can not arise by itself from the stable homogeneous regime, it can be realized if initial conditions are properly prepared.

A concrete type and size of the local inhomogeneity do not play a large role (see below). The only condition supposed to be satisfied is as follows: transition to chaos is not connected with the region of inhomogeneity, but it is defined by an asymptotic behavior in the homogeneous region.

Asymptotically, the solution is homogeneous at $x \to \infty$ before the transition to chaos. Consequently, period-doublings take place at the same parameter values as in the point system. However, the solution with time period 2^n, appearing at the nth bifurcation, becomes homogeneous only at a certain characteristic distance ξ_n from the local inhomogeneity. Let us call the region, where the solution differs from the homogeneous one, a tail and the value ξ_n – a tail length. The scaling relation derived in the section 3 yields $\xi_{n+1} = \beta \xi_n$. It is, therefore, clear, that there is no local similarity between components of the 2^n and 2^{n+1} cycles in the fixed space points. Universal features may appear only if we take into account scale change along the x-axis. Note also, that $\xi_n \to \infty$ at $n \to \infty$, so the tail becomes much larger than the size of inhomogeneity and the latter does not play a role.

These considerations suggest a hypothesis of the tails similarity. For simplicity, we assume first that the parameter is equal to the critical value λ_c^o. We'll describe the form of the tail of the solution component with time period $N = 2^n$, using the quantity

$$S(x, n) = \frac{1}{N} \sum_{i=1}^{N} \left(u_i(x) - u_{i+N/2}(x) \right)^2, \qquad (19)$$

which is equal to the sum of the intensities of the spectral peaks, emerged at the nth doubling bifurcation. According to [16], the quantity $S(n)$ in the point system obeys the universal power law behavior

$$S(n) = \gamma S(n+1), \quad \ln S = \text{const} - n \ln \gamma,$$
$$\gamma = 10.48\ldots .$$

In the distributed system S depends both on n and x. The assumptions given above enables one to suppose that

$$S(n, x) = \gamma S(n+1, \beta x), \qquad (20)$$

or

$$\ln S(n, x) = \text{const} - n \ln \gamma + \theta\left(\frac{x}{\Delta} \beta^{-n} \right). \qquad (21)$$

The function $\theta(y)$ should be universal (at least, for large y). Indeed, the solution for large y is weakly inhomogeneous and the form of the tail is determined by the universal operator G. The scaling relation is generalized for the case $\lambda \neq \lambda_c^0$ in the following way: if the value $S(n, x)$ is calculated at some λ_1, then in the right-hand side of the formula (20) there should be a value $S(n+1, \beta x)$ for $\lambda = \lambda_c^0 + (\lambda_1 - \lambda_c^0)\delta^{-1}$.

We expect the formulated similarity relation to be valid beyond the critical point too. Here the role of S is played by the total intensity of the broad-band part of the power spectrum $S(\lambda, x)$. Since the broad-band spectrum results from the discrete spectrum disappearance, its scaling may be written down similarly to (21):

$$S(\lambda - \lambda_c^0, x) = \gamma S\left(\frac{\lambda - \lambda_c^0}{\delta}, \beta x \right). \qquad (22)$$

Clearly, at large distances from the local inhomogeneity, the statistically homogeneous state is realized. Formula (22) shows that this homogeneity is disturbed at the distance of the order of the correlation length (18).

4.2. *The finite system*

We consider now the system with finite length $0 \leq x \leq L$. For simplicity, we assume that there is a local inhomogeneity (for example, $u(0)$ is fixed) at the left boundary, and at the right boundary free boundary conditions are imposed. This configuration can be considered as a half of the symmetrical system with length $2L$, which is inhomogeneous at both boundaries. Note that the delay systems, described in section 1, are always finite.

The structure of bifurcations in the finite system is as follows. If the system length is large compared with the diffusion length ($L \gg \Delta$), then the tails near the left boundary are formed during the first few period-doubling bifurcations, and the solution is practically homogeneous near the right boundary. Therefore, the bifurcation points and the spatial configuration of the solution are the same as in the semi-infinite system with the local inhomogeneity. With the increase of the number of bifurcations n, the tail length increases proportionally to β^n. Consequently, at some n it becomes comparable with the system length: $\xi_n \simeq L$. This results in the interaction of the tail with the right boundary, and in the disturbance of the bifurcation values of the parameter. The tail form is universal according to the similarity hypothesis (21). Therefore, the disturbance of the bifurcation points should also be universal and is determined, roughly speaking, by the value of the tail at the boundary. Proceeding from this, we formulate the similarity hypothesis for the bifurcation values of the parameter

$$\lambda_c^0 - \lambda_n = \delta^{-n} K_0 \varphi\left(\frac{L}{\Delta}\beta^n\right). \tag{23}$$

Here λ_c^0 and K_0 are constants in the Feigenbaum formula for bifurcations in the point system: $\lambda_n^0 = \lambda_c^0 - K_0\delta^{-n}$, φ is the universal function. Since the bifurcations in the infinite system occur at the same parameter values as in the point one,

$$\varphi(z) \to 1 \text{ at } z \to \infty.$$

With the further increase of n tails cease to exist because their length exceeds the system length. The only characteristic spatial scale L remains. Therefore, the spatial distribution of the large n spectral components becomes fixed and independent of n. The system is effectively non-distributed in this domain, and the period-doubling bifurcations accumulate to some critical value λ_c, obeying the ordinary Feigenbaum law

$$\lambda_c - \lambda_n = K\delta^{-n}. \tag{24}$$

The relation (24) can be brought into accordance with (23) if we suppose that at $z \to 0$ $\varphi(z) \to -Az^{-\kappa} + B$, where $\kappa = \nu^{-1} = 4.44\ldots$. The constants κ, A, B are expected to be universal, since the function $\varphi(z)$ is also universal. Substituting this relation into (23), we obtain the connection between λ_c, K and λ_c^0, K_0:

$$\lambda_c = \lambda_c^0 + AK_0\left(\frac{L}{\Delta}\right)^{-\kappa}, \tag{25}$$

$$K = BK_0. \tag{26}$$

In the supercritical domain $\lambda > \lambda_c$ the finite system behaves as a non-distributed one, as long as the correlation length r_c given by (18) exceeds the system length L. Therewith, the dynamics is chaotic in time but the spatial distribution remains fixed. With the further increase of λ r_c decreases, and the point where $r_c \simeq L$ is exceeded. Then the picture of the spatial distribution of the chaotic motions becomes the same as in the infinite system with the local inhomogeneity.

5. Numerical results

In order to verify the scaling hypotheses formulated in the previous section, we have investigated numerically the following discrete system:

$$u_{n+1}(x) = \lambda\left(1 - 2u_n^2(x)\right) + D\left(u_{n+1}(x-1) - 2u_{n+1}(x) + u_{n+1}(x+1)\right). \tag{27}$$

Eq. (27) is, evidently, of the type (5), in which $f(u,\lambda) = \lambda(1-2u^2)$ and the Fourier transform of the kernel of the linear operator $\hat{m}$ has the form

$$m(k) = \frac{1}{1 + 2D(1-\cos k)}. \tag{28}$$

It may be easily verified that eq. (27) satisfies all the conditions given in section 2. The diffusion length Δ is equal to $\sqrt{2D}$. The system length used in the calculations was $L = 100$, boundary conditions were as follows:

$$u(0) = 0, \quad u(L+1) = u(L). \tag{29}$$

For different values of D within the interval 0.5–1000 the bifurcation points of λ were found

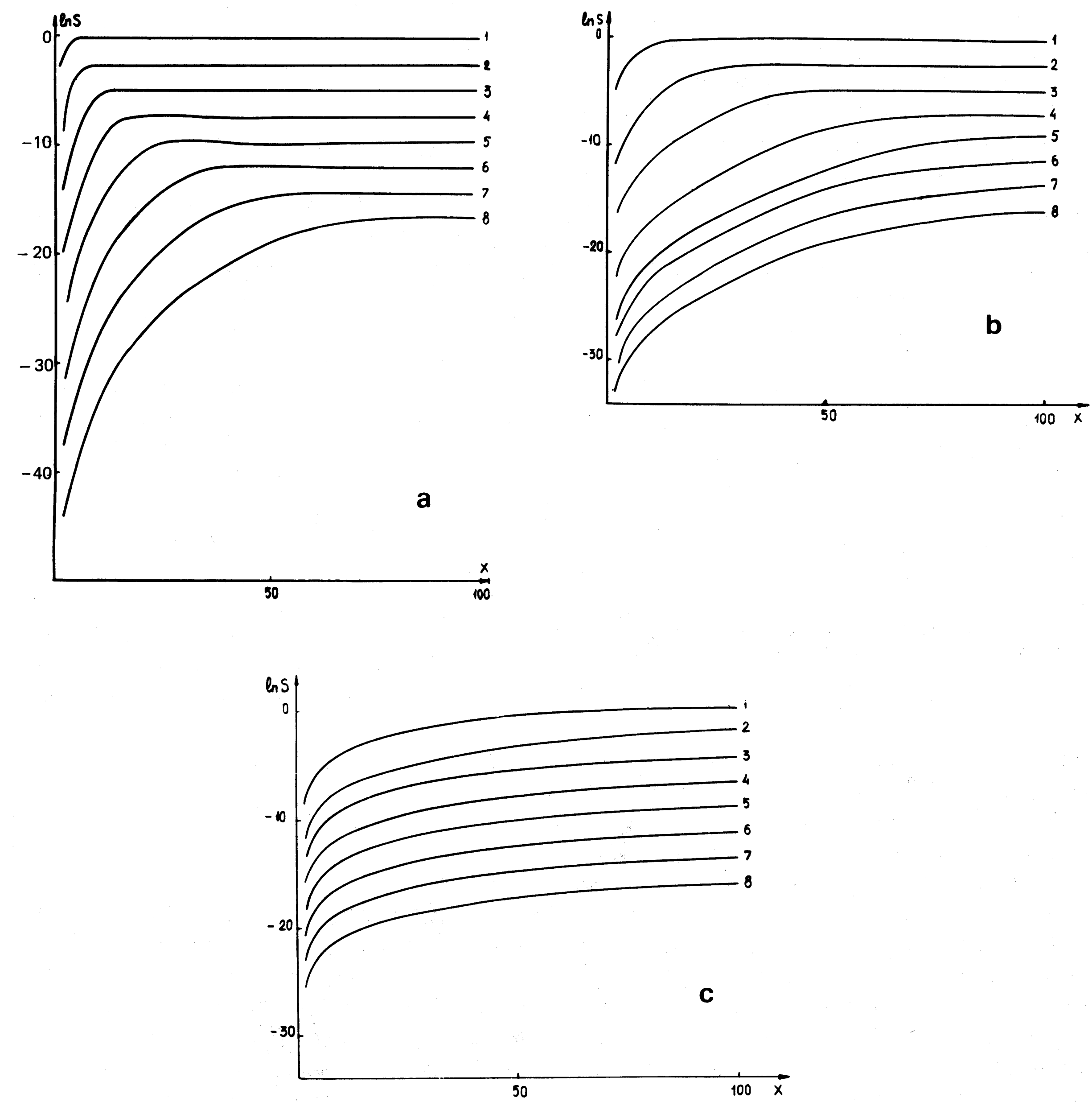

Fig. 1. The functions $\ln S(n, x)$ for a) $D = 1$; b) $D = 20$; and c) $D = 1000$.

for the cycles of the periods $1, 2, 4 \ldots, 256$. The values $S(n, x)$ defined by (19), which characterize the intensity of the 2^n components in the point with coordinate x, were also calculated.

Fig. 1 shows the distributions of $S(n, x)$ along the system for $D = 1, 20, 1000$. The value of λ for these calculations was chosen in the domain of stability of the 512-cycle. For $D = 1$ one can vividly see the increase of the tail length with growth of n. The interaction of the tail with the right boundary for $n \lesssim 9$ is weak and the shape of the tail is practically the same as in the semi-infinite system.

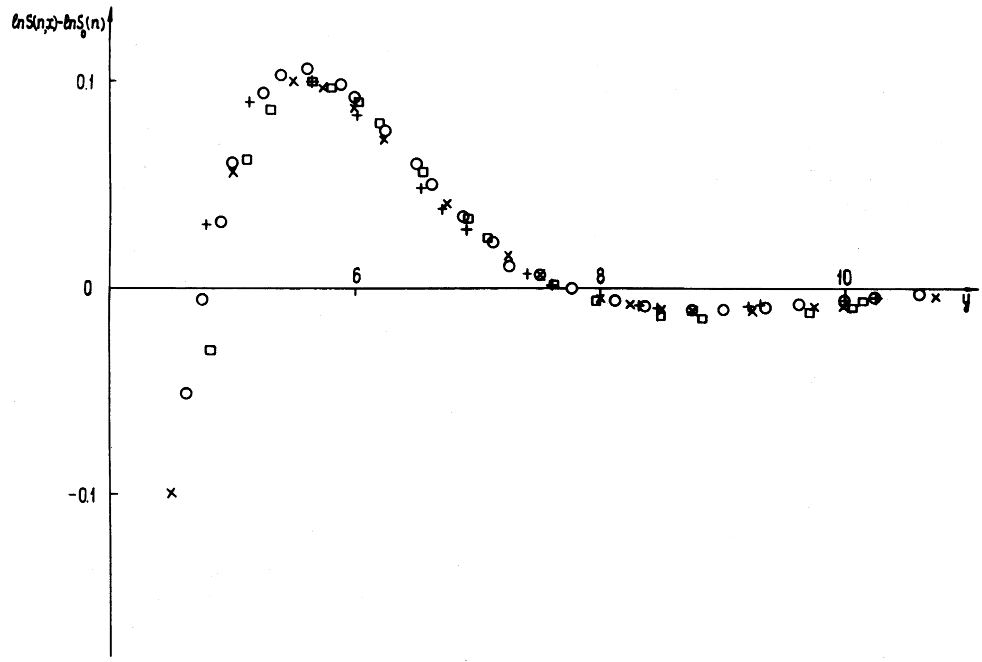

Fig. 2. The shape of the tail: the quantities $\ln S(n,x) - \ln S_0(n)$ are plotted versus $y = xD^{-1/2}\beta^{-n}$. (○) $D=1$, $n=6$; (+) $D=1$, $n=5$; (□) $D=0.5$, $n=6$; (×) $D=0.5$, $n=15$.

For $D=20$ the length of the tail becomes comparable with L at $n \simeq 4$. The region $n = 4$–6 is transient to a new regime, where the shape of the $S(n,x)$ becomes fixed and independent of n. For $D=1000$ the shape of $S(n,x)$ is practically fixed already for $n \gtrsim 2$.

Clearly, one should use the tails, obtained for small D, in order to check their similarity in the semi-infinite system (formula (21)). Fig. 2 shows the dependence of $\ln S(n,x) - \ln S_0(n)$ on the coordinate $y = xD^{-1/2}\beta^{-n}$ (here $S_0(n)$ refers to the point system). One can see from fig. 2 that for large n, the points corresponding to different n and D fit the same curve, which is, obviously, the plot of the universal function $\theta(y)$. It must be mentioned that the dispersion of the points can be diminished, if one uses the shift of the centre of the scaling, i.e. takes the variable $y = (x + x_*)D^{-1/2}\beta^{-n}$.

Fig. 3 shows the relation of the intensities of the components with period 2^n and 2^{n+1} at the right boundary as the function of n and D:

$$q(n,D) = \frac{S(n,L)}{S(n+1,L)}.$$

One can readily distinct three regions: 1, 2, 3. In region 1 the solution near the boundary is practically homogeneous and the value q is close to the universal constant γ. The region 2 corresponds to the interaction of the tail with the boundary, here the declination of values q from γ is observed. Finally, in region 3 the system behaves like a non-distributed one and q is again close to γ.

Now we turn to the similarity hypotheses for the bifurcation parameter values. We start with checking the relation (25). Critical values λ_c were determined for different D by the extrapolation of the bifurcation points λ_n. The data obtained are presented in fig. 4. As it was expected, the points are arranged along the straight line, whose slope is determined by the constant κ. Note that one can

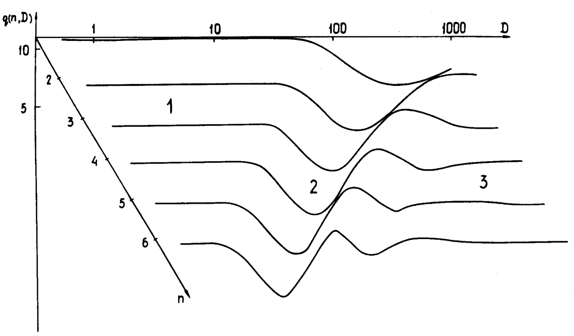

Fig. 3. The function $q(n, D)$.

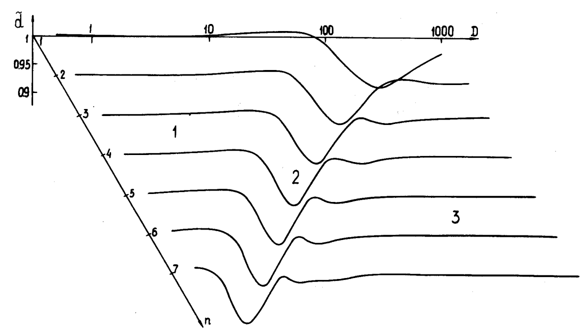

Fig. 5. The normalized bifurcation values ratio $\tilde{d} = d/d_0$, where d_0 is the corresponding ratio for the point system, as the function of n and D.

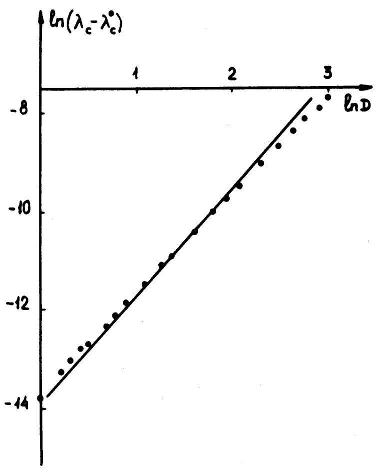

Fig. 4. The disturbance of the critical point versus the coupling constant. The line corresponds to the law (25).

improve the conformity with (25) using $L_{\text{eff}} = L + \Delta x_*$ instead of L, i.e. taking into account the centre of similarity shift, mentioned above. Using the value of K_0 for the mapping (1) ($K_0 = 0.22$) and the data presented in fig. 4 one can calculate the constant $A = 880$.

Fig. 5 gives a visual representation of the bifurcation structure in the finite system. Here the quantity $d = (\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n)$, which may be found directly from experiments is presented as the function of the parameters n and D. One can see the same regions as in fig. 3: 1) bifurcation points are close to those in the point system; 2) the region of tail boundary interaction, here the distortion of d from the Feigenbaum's constant δ is the greatest; 3) the bifurcation points satisfy the relation (24). To verify the scaling relation (23) the data should be processed suitably to distinguish the region 2, where the differences form the Feigenbaum law are large. The first method consists in plotting (see fig. 6)

$$V = \Phi(W), \tag{30}$$

where

$$W = \left(\frac{\lambda_c - \lambda_c^0}{\lambda_c - \lambda_n^0} \right)^{2/\kappa}, \quad V = \frac{\lambda_c - \lambda_n - \lambda_c^0 + \lambda_n^0}{\lambda_c - \lambda_n^0}$$

and the quantities with index "0" refer to the point system. The function $\Phi(W)$ relates to $\varphi(Z)$ in the following way:

$$\varphi(Z) = 1 - AZ^{-\kappa} + (1 + AZ^{-\kappa}) \times \Phi\left[(1 + Z^{\kappa} A^{-1})^{-2/\kappa} \right]. \tag{31}$$

Bifurcation values, plotted in the coordinates V, W, fit a distinct curve (fig. 6). The data processing using V and W is convenient for two reasons: first, unlike $\varphi(Z)$, the function $\Phi(W)$ is defined on the interval [0, 1] and has no singularities: second, it is sensitive to the deviations from the value δ in region 2 of fig. 5. Finally, we note that using (23), (31) and fig. 6 one can determine the constant B (see (26)):

$$B = 1 - \frac{2}{\kappa} \Phi'(1) \simeq 1.08.$$

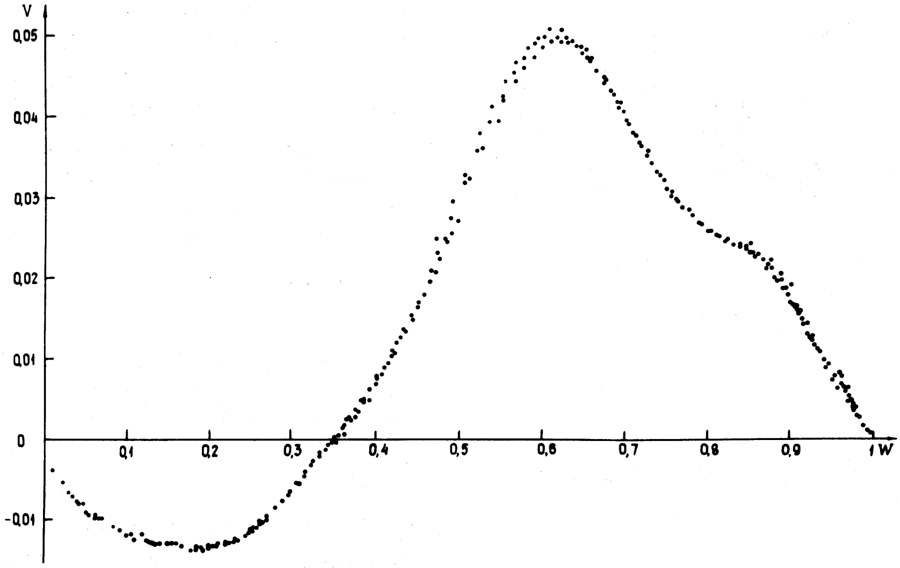

Fig. 6. The bifurcation values data obtained for different n and D, processed in variables V, W.

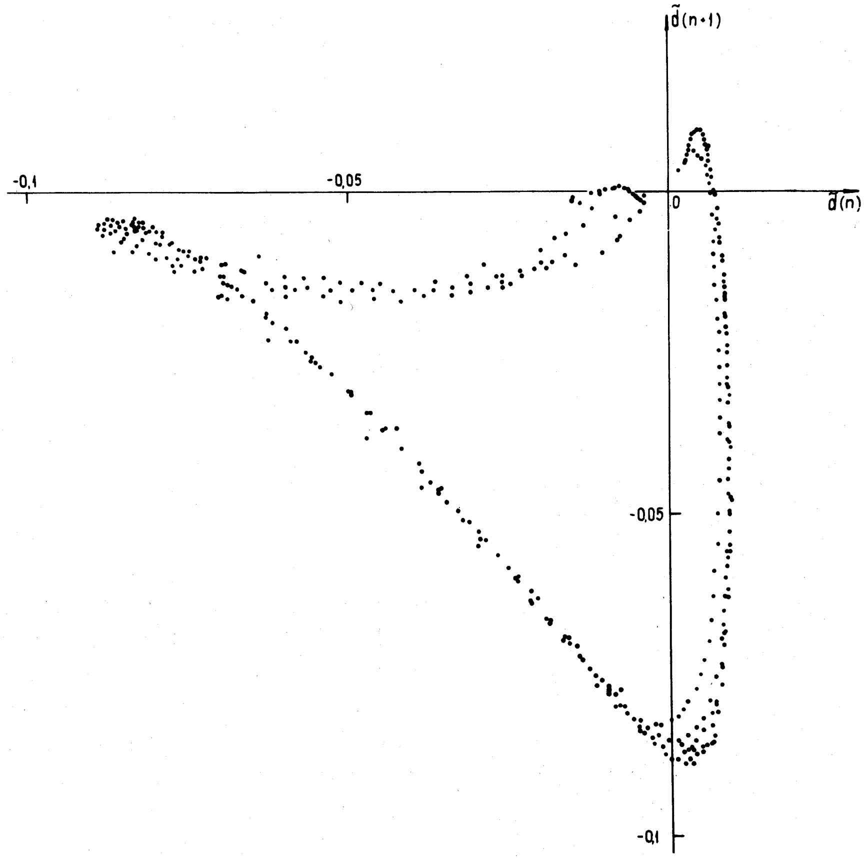

Fig. 7. The "plot of successions" for the normalized bifurcation values ratio $\tilde{d}(n)$ obtained for different D.

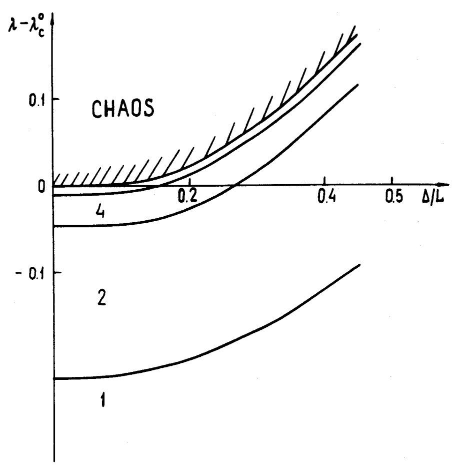

Fig. 8. The bifurcation diagram for the transition to chaos in the finite system.

However, the drawback of the above method is as follows: in order to determine the quantities V, W one has to know bifurcation points for the point system as well as the critical point λ_c. There exists another method of data processing, free from this drawback. Here only the relations $d(n)$ are used, which are plotted on the plane $(d(n), d(n+1))$. The points again fit some curve (fig. 7) and this fact shows that there is a unique universal function, which governs the values of bifurcation points disturbances.

It seems useful to draw a picture of regions with different dynamics on the plane of physical parameters (λ, L). It follows from the above considerations that this picture will be universal if we use normalized variables $(\lambda - \lambda_c^0)K_0^{-1}$ and Δ/L. The picture is presented in fig. 8. Note the scale invariance of fig. 8 near the origin – any curve converts into itself if the scale along the $(\lambda - \lambda_c^0)$ axis is multiplied by δ and along the Δ/L-axis by β.

Besides the model (27) with boundary conditions (29) we investigated numerically some discrete systems of the type (5), which differ by the choice of the function $f(u, \lambda)$, of the operator $\hat{m}$ and of the boundary conditions. In particular, the case of free boundary conditions with local inhomogeneity in λ such as $\lambda(x) = \lambda[1 - \exp(-x^2/L_0^2)]$ was investigated, as well as the system with boundary conditions $u_{n+1}(0) = u_n(1)$, $u_n(L+1) = u_n(L)$, corresponding to the phase dislocation. We obtained in these cases the quantitative results, close to those described above, providing support for the proposed scaling and universality of the functions $\theta(y)$ and $\varphi(Z)$.

6. Conclusion

Let us discuss the results obtained from the viewpoint of the general problem of transition to chaos in distributed systems. First, our results revealed a peculiar property of the distributed systems: the evolution of spatial structures in the course of the transition to chaos. The basic idea used in this paper is the construction of the distributed medium using coupled point systems with the known universal properties. Several types of universal critical behavior have been described recently [2, 17–20]. Thus, one can consider distributed systems with different types of the universal transition of their elements to chaos. We note that the dissipative type of coupling is possible in each case. Therewith, our approach, based on the renormalization group for the operator of evolution, also holds. Indeed, (15) shows that the scale factor β depends only on the time renormalization factor. Let the time scale increase by μ at the renormalization, then $\beta = \sqrt{\mu}$ for dissipative coupling. Consequently, if, for example, the transition to chaos takes place via variation of some parameter Λ characterized by the scale factor δ_Λ, then the spatial correlation length in the dissipative medium changes according to the power law $r_c \approx (\Lambda - \Lambda_c)^{-\chi}$, where $\chi = \log\beta/\log\delta_\Lambda$.

In conclusion we'd like to note that the problem of the period-doubling transition to chaos in the medium of coupled nonlinear elements is not exhausted here, since the dissipative coupling is not the only one possible. It was shown in [21] that during period-doublings the coupling defined by smooth functions of phase variables tends to the combination of two universal types of the coupling. These types have different transformation properties under the renormalization. One of them corresponds exactly to the dissipative coupling considered here, while the other type provides the possibility of the emergence of the inhomogeneous states in the infinite homogeneous system already at $\lambda < \lambda_c^0$, thus the scenario of transition to chaos becomes complicated.

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