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CHAOTIC AUTOWAVES IN EXCITABLE MEDIA*

A. S. PIKOVSKII

Institute of Applied Physics, U.S.S.R. Academy of Sciences, Gorkii

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In a simple model the author demonstrates different regimes of chaotic autowaves in an excitable medium with diffusion. A wave of twice the period appearing in a bistable medium is diescribed. The mechanism associated with diffusion of stabilization of unstable chaotic oscillations is discussed. The possible existence of a solitary chaotic wave is shown.

AUTOWAVE processes in active distributed systems with diffusion have recently been widely discussed in connexion with the problems of biophysics, chemical kinetics, ecology, physics, etc. (see reviews [1–31]). Most models of one-dimensional excitable media are set by the parabolic equations

$$\partial u_i/\partial t=f_i(u_1, \ldots, u_N)+D_i\partial^2 u_i/\partial x^2, \tag{1}$$

where the variables u_i have a meaning differing with the applications (population size in ecology, concentrations of substances in chemical kinetics, etc.); D_i are diffusion coefficients; f_i are non-linear functions describing the dynamics of a point system.

The types of possible autowaves are essentially determined by the behaviour of the point system. For example, if in a point system there is bistability, i.e. two stable states of equilbrium exist, then an autowave in the form of a jump is observed in the distributed system.

If a point system has only one stable equilibrium state but in response to an external perturbation a pulse of finite duration is generated, then in a distributed system a solitary steady autowave may spread.

This work describes a new class of autowaves—chaotic autowaves. They appear if the point system has chaotic oscillations which corresponds to movement on a singular attractor in phasic space (see review [4]). Such regimes have been observed in a number of numerical investigations of multidimensional point systems of ecology, chemical kinetics, etc. (see, for example, [5, 6]). There is much less experimental work, and we would note here the reliably established stochastic regimes in the Belousov–Zhabotinskii homogeneous reaction [7].

As with regular autowaves one may isolate two types of chaotic autowaves—in the form of a jump and solitary ones. The chaotic autowaves in the form of a jump are

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possible if in the point system generalized bistability is observed of the type "stable equilibrium state–chaos" (or "chaos–chaos", "periodic regime–chaos"). In space there exist two infinite regions with different type of behaviour separated by a moving jump. Of special interest is determination of the dynamics of the jump itself the rate of which may be not constant.

A chaotic solitary autowave is possible if in the point system together with the stable equilibrium state there is a metastable chaotic multiplicity. Then in response to a finite perturbation the point system in the course of a certain time interval generates chaotic oscillations. Taking diffusion into account these oscillations begin to take in adjacent regions and the autowave will spread in the form of a packet of stochastic auto-oscillations.

Discrete model of an excitable medium. As the basic model for numerical investigation of chaotic autowaves we chose a system discrete in time and space

$$u_i(n+1,j)=F_i(u_1(n,j),\ldots u_N(n,j)) + D_i(u_i(u+1,j-1)-2u_i(n+1,j)+u_i(n+1,j+1)). \quad (2)$$

Here, $n=0, 1, 2, \ldots$ is discrete time; $j=\ldots, -1, 0, 1, 2\ldots$ is a discrete coordinate. The point system obtained from (2) at $D_i=0$ is an N-dimensional reflexion. It may be interpreted as the reflexion of the shift per unit time for a point system of differential equations (1). Discrete models are also often used to describe different ecological systems [8, 9]. We would emphasize that system (2) should not be regarded as a discrete approximation of the equations (1). In particular, the functions F_i must not be directly related

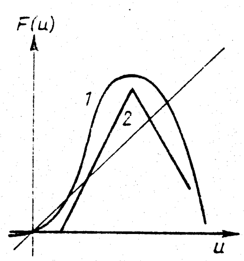

FIG. 1

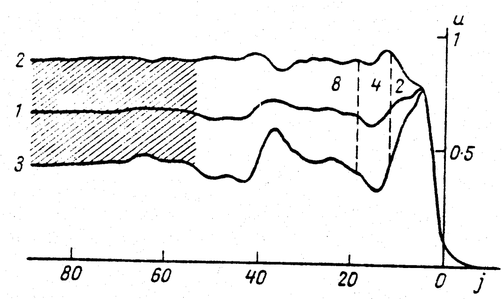

FIG. 2

FIG. 2. Structure of an autowave of the jumps type at $D=2$; $a=3{\cdot}9$; $b=20$; $u_0=0{\cdot}1$. Curve *1* – average profile; curves *2* and *3* are separated by the standard deviation. In the shaded region the oscillations are chaotic.

to the functions f_i. In this sense (2) is an independent model of an excitable medium. Its use is justified by the fact that ordinary differential equations can often be reduced (analytically or numerically) to discrete reflexions. In addition, numerical experiment with model (2) is very effective since on discrete approximation of (1) it is necessary to perform 10–20 iterations for the characteristic period of the oscillations whereas in model (2) they are replaced by one iteration.

Chaotic autowaves in the form of a jump. To obtain a chaotic wave in the form of a jump one variable u suffices. Equations (2) were solved numerically in the region of a finite but sufficiently large length so that the influence of the boundaries was not manifest. Depending on the type of function $F(u)$ we obtained autowaves of different structure.

1. The reflexion $F(u)$ had the form of curve *1* in Fig. 1. The specific calculations were made with the reflexion:

$$F(u)=au(1-u)/[1+\exp(-b(u-u_0))]. \tag{3}$$

It has two stable regimes—zero equilibrium state and peculiar attractor. In fixing the initial jump ($u=0$ for $j<j_0$, $u=u_1$ for $j>j_0$ where j_0 is the initial coordinate of the jump the value u_1 lies in the region of the peculiar attractor) a spreading autowave forms. Numerical calculations showed that the speed of propagation of the jump is constant. This, at first sight paradoxical, result is explained by the structure of the wave formed. To investigate this structure, the instant profiles of the autowave were so combined that they coincided at the point $u=u_0$. As a result we found the mean profile and dispersion (Fig. 2). The instant profile does not coincide with the mean but changes chaotically only far from the front. Close to the front the oscillations of the profile are periodic and the period doubles on moving away from the jump (see Fig. 2).

At a sufficient distance from the front ($j\gtrsim 70$) statistically uniform chaotic oscillations are observed with the complex structure of the front having practically no influence on them. Such a wave may be called a wave of the double period. At each point in space stochasticity develops but not simultaneously with the arrival of the front of the wave, which also governs the constancy of speed. The resulting structure of doublings is apparently associated with the form of the point reflexion (3) in which the transition to chaos with increase in the parameter a occurs through bifurcation of the doubling of the period. A somewhat different structure of the wave is obtained for the bit-linear reflexion (curve *2* in Fig. 1). Here, the speed of the wave is also practically constant and close to the front the profile oscillates with the period 2. However, these oscillations quite rapidly give way to chaos and in the transitional region have the form of periodic oscillations with the period 2, on which noise is superposed.

The regular structure of the wave front may be explained by the following factors. Firstly, the oscillations at each point of space begin from the same zero level. If the degree of chaotization is low, then in the period of traversal of the front of the wave stochastization does not have time to occur. Secondly, the neighbourhood with an unexcited region is equivalent to additional dissipation which reduces the level of chaos in the reflexions of the type in Fig. 1.

2. Spread of the chaotic autowave is also possible when in the point system there is only a peculiar attractor and the zero equilibrium state is unstable. Such a regular autowave corresponds to the known solution of the equations of Kolmogorov, Petrovskii and Piskunov [1]. Here, as $F(u)$ we took the so-called logistic reflexion very popular

in study of population dynamics [8]:

$$F(u)=au(1-u).$$

In this case the speed of the front was also practically constant but the structure of doubling was not observed. Apparently, this is related to the fact that the unexcited region before the front of the wave introduces too low additional attenuation.

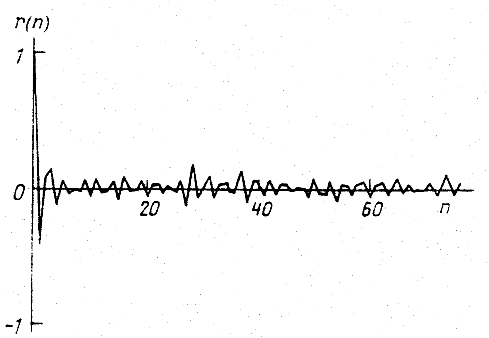

FIG. 3

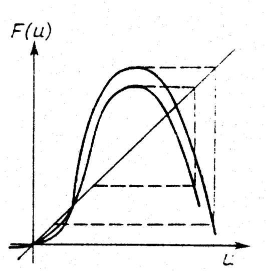

FIG. 4

3. As noted, stochastization of the speed of the front may be expected for a high mixing rate in a point system. In fact, such regimes were observed for the reflexion

$$F(u)=[1+2u_0+\cos(au)]/[1+\exp(-b(u-u_0))]. \tag{4}$$

Here for large values of a the magnitudes $|F'(u)|\sim a$, i.e. there is rapid stochastization of the movements. As a result the profile changes chaotically over the whole region and the speed of the wave undergoes considerable fluctuations with their correlation function indicated in Fig. 3.

Diffusion-induced stabilization of unstable chaotic oscillations. If in a point system set by the reflexion (4) the parameter a is increased then for a certain value a_c the peculiar attractor loses stability (a so-called crisis occurs [10], see Fig. 4). At $a>a_c$ in the point system only the zero equilibrium state is stable but chaos is metastable [11], i.e. it is observed during a limited (though possibly long) time interval after which it is disrupted.

Let us consider a regime appearing in such a system having regard to diffusion. The chaotic oscillations in a system with diffusion become heterogeneous [12, 13]. Therefore, the "disruption" will occur not over the whole region simultaneously but in certain limited portions. But since in the adjacent regions chaotic movements continue, two autowaves of the fall type form and begin to spread into the unexcited region and absorb it. Then the disruption begins at another site and so forth. As a result, a statistically uniform regime of steady chaotic oscillations may be established. This regime is known to be stable if the lifetime of the chaotic oscillations in the point system is long. It may be expected that with fall in the life time a threshold will appear below which only a trivial regime is stable.

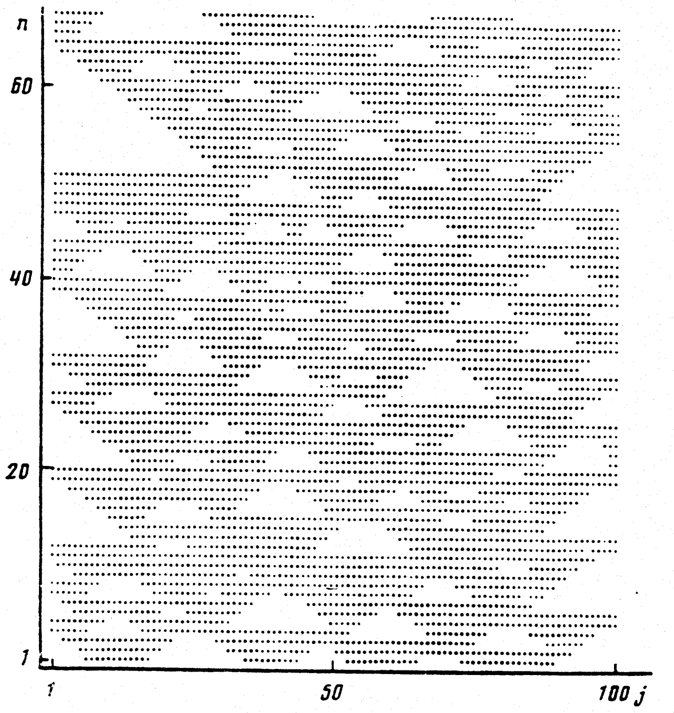

FIG. 5. Pattern of the excited regions ($u \geqslant u_0$) on the plane (n, j) for the reflexion (4) at $a=5$; $u_0=0{\cdot}2$; $D=5$; $b=20$.

The picture described was observed in the numerical experiments with the reflexion (4) and is shown in Fig. 5. It was found that the mean density of the excited regions within wide limits does not depend on the diffusion coefficient. In fact, the task has three scales of length: internal $-l=1$ is the scale of discretization; external $-L$ is the size of the region; and $\Delta \sim D^{\frac{1}{2}}$ is the characteristic size of the inhomogeneity. If $l \ll \Delta \ll L$ then the internal and external scales play no role and the structures appearing for different values of Δ are similar. If $\Delta \sim L$ then the chaotic oscillations may be disrupted simultaneously over the whole region; if $\Delta \sim l$ then the oscillations of adjacent cells are independent; in both cases a trivial regime is established.

Chaotic running pulses. Chaotic running pulses may be obtained in a two-component reflexion of type (2). It is easy to construct the corresponding model by using the qualitative ideas of running steady pulses in relaxation systems [1, 2]—it is necessary to add to the equation describing the movement of the chaotic jump an equation describing the evolution of a slow variable; for a slow variable diffusion need not be taken into account. In particular, in the numerical calculations the following reflexion was used:

$$F_u(u,v)=au(1-u)/\{1+\exp[-b(u-u_0-v)]\},$$
$$F_v(uv)=v+\varepsilon(u-v).$$

For $\varepsilon \ll 1$ the variable v evolves slowly resulting in the formation of the chaotic jump described in section 2. Far from the jump the variable v rises and the chaotic fluctuations become metastable, here the regime resembles that described in section 3. With

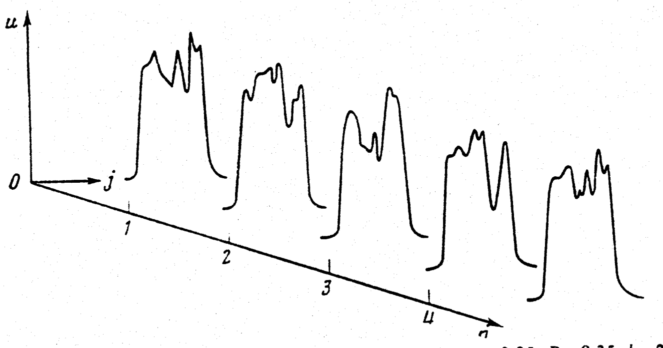

FIG. 6. Chaotic autowave in system (5) at $a=3\cdot 9$; $u_0=0\cdot 1$; $\varepsilon=0\cdot 05$; $D=0\cdot 35$; $b=20$.

further increase in v the fluctuations of the rapid variable are disrupted and a long refractory "tail" of the slow variable remains behind the pulse. The dynamics of the pulse is shown in Fig. 6. The trailing edge of the rapid variable undergoes the greatest fluctuations and together with it, the duration of the pulse.

CONCLUSION

This work has considered stochastic regimes in an excitable medium with diffusion due to the chaotic nature of the point system. We would note that other mechanisms of stochastization of distributed systems are possible associated with desynchronization of the regular oscillations in individual point subsystems [1].

Numerical modelling was undertaken with discrete reflexions. Such models come halfway between systems of parabolic equations (1) and so-called axiomatic models in which not only are time and space discrete but also the state of the system. The reflexions possess the following advantages: 1) the point reflexions reproduce all the regimes which may be observed in systems of ordinary differential equations; and 2) numerical investigation of the reflexions is simpler by an order than integration of equations (1). Therefore, it appears promising to use reflexions with diffusion of type (2) in study of chaotic and regular autowaves in two- and three-dimensional media.

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