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Discrete-Time Dynamic Noise Filtering *

A.S. PIKOVSKIY

A method for dynamic noise filtering is proposed. It is shown that such a method can be used to convert a sample obtained in a noisy system to a new sample with a lower effective noise.

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Dynamic deterministics systems are widely used as models of physical objects. Increased interest in such systems arose recently in connection with the discovery of external attractors [1, 2]. It turned out that deterministic models can be used to describe regular as well as

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irregular and quasi-random processes. An important problem in this field is the construction of a dynamic model directly from measured data. The solution of this problem has had some success [3-5]. The basic difficulty, however, is the fact that physical objects are not strictly deterministic due to the presence of fluctuations (noise). This gives rise to two problems: 1) the detection of noise — the detection of its presence in a physical system and the estimation of its level, 2) the filtering of noise — extraction of the deterministic part of the process. Methods of solving the first problem, i.e., the detection of noise, were described in [6, 7].

This paper proposes a method of filtering of noise in dynamic systems. Using a sample obtained in a noisy system, the method enables one to construct a new sample in which the effective noise is lower. The method can also be useful when determining the nature of an observed irregular behavior. Actually, an external random behavior can be observed in two situations: 1) when an external attractor is present in a completely deterministic system; 2) if the system has a regular steady state which is perturbed by fluctuations. The reduction in the effective noise by filtering enables one to distinguish between these two situations.

Let us consider a discrete process, described by a mapping which is specified by arbitrary nonlinear functions F^i and G^i:

$$X^i(n+1)=F^i(\mathbf{X}(n))+G^i(\mathbf{X}(n))\xi^i(n). \tag{1}$$

Here $\mathbf{X}=\{X^1,\dots,X^m\}$ is an m-dimensional vector which depends on discrete time n, and $\xi^i(n)$ is a sequence of independent, random quantities. We shall assume that

$$\langle\xi^i\rangle=0,\quad \langle\xi^i(n)\xi^i(l)\rangle=\delta(n-l). \tag{2}$$

Suppose that we have a fairly long sample of the process, i.e., the values of $\mathbf{X}(n)$ are known for $n = 1, \dots, N+1$. Let us define a new sample $\mathbf{Y}(k)$ by the following recursion formula:

$$Y^i(k+1)=\frac{\sum_{j=1}^{N} s(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)X^i(j+1)}{\sum_{j=1}^{N} s(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)}. \tag{3}$$

Here $\varepsilon \ll 1$ is a small parameter and $s(u)$ is a weighting function which decreases fairly rapidly as $u \to \infty$ (in all examples below we use the weighting function $s(u) = \exp(-u^2)$). Formula (3) has a simple meaning: to obtain a mapping of the point $\mathbf{Y}(k)$, we average over all points whose original mappings are located in the ε-neighborhood of the point $\mathbf{Y}(k)$. Substituting (1) into (3), we obtain:

$$Y^i(k+1)=\bar{Y}^i(k+1)+\tilde{Y}^i(k+1),$$

where

$$\bar{Y}^i(k+1)=\frac{\sum_{j=1}^{N} s(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)F^i(\mathbf{X}(j))}{\sum_{j=1}^{N} s(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)}, \tag{4a}$$

$$\tilde{Y}^i(k+1)=\frac{\sum_{j=1}^{N} s(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)G^i[\mathbf{X}(j)]\xi^i(j)}{\sum_{j=1}^{N} s(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)}. \tag{4b}$$

The transformation $\mathbf{Y}(k)\to\mathbf{Y}(k+1)$ can be treated as a mapping in which $\bar{Y}^i(k+1)$ describes its regular part and $\tilde{Y}^i(k+1)$ describes the imaginary part. Let us derive expressions for these two parts of the mapping in the limit as $N \to \infty$ and $\varepsilon \to 0$. When $N \to \infty$ and $\mathbf{X}(n)$ is an ergodic process, the summation in (4) can be replaced by an integral over the invariant probability density

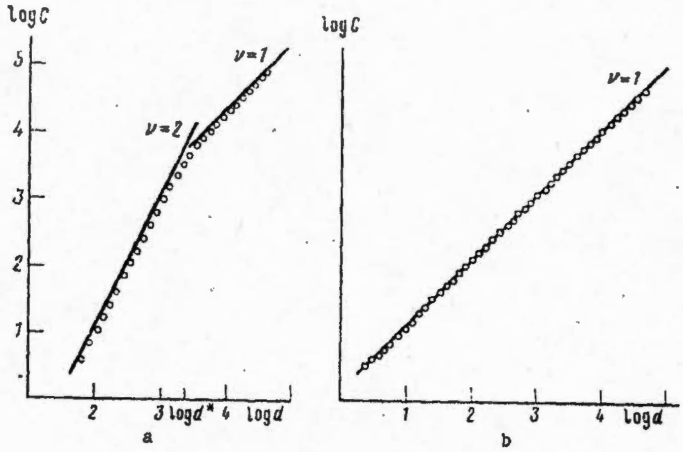

Fig. 1. Detection of noise proposed in [7, 8]; a) before filtering the function $C(d)$ has a break. When $d > d^*$, the attractor is almost univariate ($\nu \approx 1$), and for $d < d^*$ it is smeared out by noise ($\nu \approx 2$); b) after filtering the attractor is practically univariate over the entire range of d.

function $W(\mathbf{X})$:

$$\bar{Y}^i(k+1)=\frac{N\int s(\varepsilon^{-1}|\mathbf{X}-\mathbf{Y}(k)|)W(\mathbf{X})F^i(\mathbf{X})d^m\mathbf{X}}{N\int s(\varepsilon^{-1}|\mathbf{X}-\mathbf{Y}(k)|)d^m\mathbf{X}W(\mathbf{X})}. \tag{5}$$

When ε is small, we expand $W(\mathbf{X})$ and $F^i(\mathbf{X})$ in a Taylor series about the point $\mathbf{X} = \mathbf{Y}(k)$, obtaining the following from (5):

$$\bar{Y}^i(k+1)=F^i(\mathbf{Y}(k))+\varepsilon^2\frac{ca^{-1}}{W(\mathbf{Y}(k))}\sum_{j=1}^{m}\left(W_jF_j^i+\frac{1}{2}WF_{jj}^i\right). \tag{6}$$

Here $a=\int s(|u|)d^mu$, $c=\int (u^i)^2s(|u|)d^mu$, W_j, F_j^i, and F_{jj}^i are partial derivatives ($W_j=\partial W(\mathbf{Y}(k))/\partial Y^j(k)$, etc.). It is seen from expression (6) that as $\varepsilon \to 0$ the regular part of the mapping $\mathbf{Y}(k)\to\mathbf{Y}(k+1)$ is identical with the deterministic part of expression (1). Formula (6) also gives the correction due to the finite value of ε.

Let us now consider the random part of mapping $\tilde{Y}^i$. In view of (2), its mean value is obviously equal to zero:

$$\langle\tilde{Y}^i(k+1)\rangle=0,$$

and the variance is given by

$$\langle(\tilde{Y}^i(k+1))^2\rangle=\frac{\sum_{j=1}^{N}s^2(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)[G^i(\mathbf{X}(j))]^2}{\left[\sum_{j=1}^{N}s(\varepsilon^{-1}|\mathbf{X}(j)-\mathbf{Y}(k)|)\right]^2}.$$

Just as in (5), we replace the summation by an integral over the invariant distribution function, obtaining the following first-order approximation in ε:

$$\langle(\tilde{Y}^i(k+1))^2\rangle=[G^i(\mathbf{Y}(k))]^2\frac{b}{N\varepsilon^mW(\mathbf{Y}(k))a^2}. \tag{7}$$

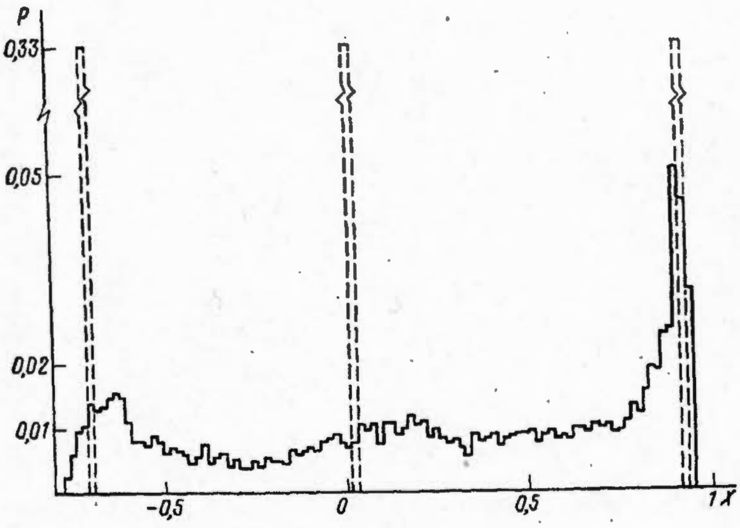

Fig. 2. Histogram P of the distribution of X: the continuous lines — before filtering, the dashed lines — after filtering.

Here $b=\int s^2(|u|)d^m u$. It is seen from expression (7) that the noise variance has decreased T-fold relative to that of original system (1), where

$$T=N\varepsilon^m W(\mathbf{Y}(k))a^2 b^{-1}. \tag{8}$$

The meaning of T is simple: apart from a numerical multiplier, it is equal to the number of points included in the effective averaging of (1).

Therefore, the new sequence $\mathbf{Y}(k)$ satisfies a discrete equation in which the effective noise is lower. In the limit as $N\to\infty$, $\varepsilon\to 0$ and $N\varepsilon^m\to\infty$, the regular part of the mapping $\mathbf{Y}(k)\to\mathbf{Y}(k+1)$ becomes identical with the deterministic part of (1) and the noise approaches zero. A finite value of ε distorts the deterministic part of the mapping described by (6) and a finite value of $T\sim N\varepsilon^m$ results in some residual noise.

The method described above was applied to the univariate mapping:

$$X(n+1)=F(X(n))+r\xi(n), \tag{9}$$

where $\xi(n)$ is a sequence of independent random quantities, distributed uniformly in the interval (-0.5, 0.5). Two cases were investigated numerically.

Case 1. $F(x)=1.5(1-|x-1|)$. When $\xi\equiv 0$, this mapping demonstrates a random behavior. The external attractor is the unvariate segment $(0.75 < x < 1.5)$. The presence of noise ($r = 0.1$) which "smears" this segment into a narrow strip can be easily determined by using the Grassberger-Procaccia method [7, 8] to measure the dimensions of an invariant set (see Fig. 1 which shows the function $C(d)$, where $C=\sum_{i,j}\theta(d-|Z_i-Z_j|)$; $\theta(\cdot)$ is a Heaviside function and $Z_i=\{X_i, X_{i+1}\}$). For small d we obtain $C(d)\sim d^\nu$, where ν is the dimensionality of the attractor [8]. After filtering with parameters $N = 1.4\times 10^4$ and $\varepsilon = 0.01$, we obtained another univariate, external attractor.

Case 2. $F(x) = 0.83671705\,(1-2x^2)$. When $\xi\equiv 0$, this mapping has a stable cycle with period 3. Introduction of noise ($r = 0.04$) disturbs this cycle and produces a quasi-univariate stochastic set. The cycle of period 3 is recovered after filtering (Fig. 2, $N = 5\times 10^3$, $\varepsilon = 0.01$).

Note that the proposed method does not require the noise to be small. However, it applies to systems in which the noise is introduced in a special way according to Eq. (1). If we consider a general mapping with random parameters:

$$X^i(n+1)=F^i(\mathbf{X}(n), \xi^i(n)), \tag{10}$$

then the use of the above method gives the mapping $\mathbf{Y}(k)\to\mathbf{Y}(k+1)$, specified by the function $\bar{F}^i(\mathbf{X})=\langle F^i(\mathbf{X}, \xi^i)\rangle$ and not by the function $F^i(\mathbf{X}, 0)$. When the noise is small, the first-order approximation of mapping (10) reduces to (1) and $\bar{F}^i(\mathbf{X})$ becomes identical with $F^i(\mathbf{X}, 0)$.

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