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Synchronization of the Stochastic Self-Excited Oscillation Phase Using a Periodic External Signal*

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The effect of an external periodic signal on stochastic self-excited oscillations, which can be represented as an oscillatory process with random amplitude and phase modulation, is investigated theoretically and experimentally. The phase synchronization effect, consisting of partial suppression of the phase spread, is described.

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INTRODUCTION

Stochastic (i.e., noiselike) self-excited oscillations, whose random nature is determined by the dynamic behavior itself rather than by fluctuations, are at present found in numerous nonlinear systems. In particular, stochastic modes have been investigated in detail in

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electronic oscillators in the RF and microwave bands [1-4]. Despite the fact that the dynamic properties of such oscillators are appreciably different, they have some common characteristics. For example, in many cases stochastic self-oscillations have the form of an oscillatory process with random amplitude and phase modulation. The power spectrum then consists of a wideband pedestal and a comparatively narrow peak (see Fig. 4a). This type of oscillation includes in particular, random modes arising due to a sequence of duplications of the period.

In this paper we investigate theoretically and experimentally the effect of an external periodic signal on stochastic self-excited oscillations. It is known that when the external signal amplitude is large enough, complete discretization of the spectrum can occur, i.e., the stochastic mode changes into a regular mode [5]. We shall consider another effect which manifests itself when the external signal is small. This consists of synchronization of the stochastic self-excited oscillation phase, i.e., the partial suppression of the phase spread. Such an external signal has practically no effect on amplitude modulation. In the power spectrum, the appearance of a discrete component in the place of the narrow peak corresponds to this effect, and the wideband part of the spectrum is almost unchanged.

1. THEORY

Let us consider synchronization of the phase of stochastic self-excited oscillations in a general type of system. As a rule, a dynamic process, described by a system of ordinary differential equations, can be reduced to a discrete mapping (Poincaré mapping): the variables are noted only at those instants of time when the trajectory in the phase space intersects the secant surface Σ (see Fig. 1). It is natural then to interpret the coordinate on Σ as the amplitude vector **U**, and the motion from one intersection of Σ to another as a change in phase 1 from 0 to 2π. A noiselike amplitude modulation corresponds to the stochastic mode in Poincaré mapping. The properties of the phase modulation are determined not only by the statistical characteristics of the representation but also by the nonisochronism parameter, i.e., by the dependence of the time of return on the secant on the amplitude. In general, all autonomous systems are nonisochronous.

Let us define the phase of an arbitrary state X as follows (see Fig. 1). Let t be time over which the trajectory arrives at this state from point U on Σ; $\tau(\mathbf{U})$ is the time of complete rotation of the trajectory which begins at point **U**. Then

$$\varphi=2\pi t/\tau(\mathbf{U}).$$

Since we are interested in the action of the periodic external signal, let us introduce a mapping of the phase in terms of the fixed time interval T:

$$\varphi_{k+1}=\varphi_k+g(\mathbf{U}_k,T), \tag{1}$$

where φ_k is the phase at the time instant kT. When the interval T is less than the characteristic period, the phase excursion g is expressed in terms of the "instantaneous frequency":

$$g(\mathbf{U}_k,T)=\frac{2\pi}{\tau(\mathbf{U}_k)}T.$$

In general, one has to take into account variations in the frequency for different intersections of the surface Σ by the secant. Note that the phase excursion is independent of the surface itself and this is a consequence of the fact that the autonomous system is invariant to a time shift. Correspondingly, $d\varphi_{k+1}/d\varphi_k=1$, i.e., there is neither an extension nor a compression in phase. It follows from (1) that the stochastic nature of the variation in the amplitude U_k leads to spread of the phase as the sum of random quantities.

Let us take into account the action of a periodic signal with period T. Now the invariance to a time shift is violated and mapping (1) is modified as follows:

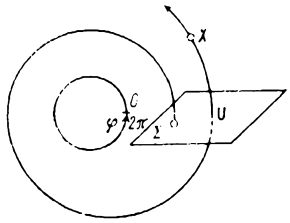
$$\varphi_{k+1}=\varphi_k+g(\mathbf{U}_k,T)+f(\mathbf{U}_k,\varphi_k). \tag{2}$$

Here the function f depends on the method of applying the external signal, its strength, etc.

Equation (2) can be simplified when the following conditions are satisfied.

1. The period T is close to the mean period of natural oscillations:

$$\langle g\rangle=2\pi+\Delta,\quad |\Delta|\ll 1.$$



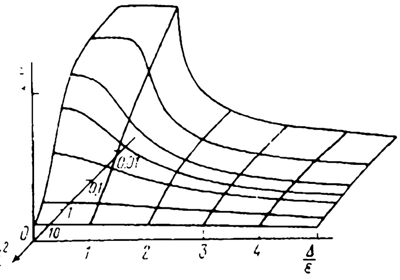


Fig. 1 Fig. 2

Fig. 1. Poincaré mapping for stochastic self-excited oscillations.

Fig. 2. Curves of the discrete component amplitude against the parameters Δ, ε, σ.

2. The degree of nonisochronism of the oscillations is small:

$$|\xi| \ll 1, \text{ where } \xi = g - \langle g \rangle.$$

3. In the expansion of the external signal

$$f(U, \varphi) = \sum_n C_n(U) \exp(in\varphi)$$

we can restrict the analysis to the first harmonic and ignore its dependence on the amplitude, i.e., we can put (with an appropriate choice of phase reference) $f(U, \varphi) = \varepsilon \cos \varphi$, $\varepsilon \ll 1$.

Thus, expression (2) can be reduced to the mapping

$$\varphi_{k+1} = \varphi_k + \Delta + \xi_k + \varepsilon \cos \varphi_k. \tag{3}$$

Since the quantities Δ, ξ, ε are small, on each step of mapping (3) the phase changes by a small amount. One can then change to a slowly varying phase 1:

$$T \frac{d\psi}{dt} = \varphi_{k+1} - \varphi_k = \Delta + \varepsilon \cos \psi + \eta(t). \tag{4}$$

Here we have replaced the random quantity ξ_k (random by virtue of the dependence of the stochastically varying amplitude) by the noise process $\eta(t)$; henceforth we shall assume it to be Gaussian and delta-correlated: $\langle \eta(t)\eta(t+\tau) \rangle = \sigma^2 \delta(\tau)$. Such a change is valid since for small values of ξ a considerable change in phase occurs after many periods. Therefore, on the one hand, the correlation time of ξ is short compared with the characteristic time of the process and, on the other hand, the increment of the phase as the sum of a large number of random quantities can be assumed to be Gaussian. The parameter σ is defined by the phase spread coefficient

$$\sigma^2 = T \lim_{N \to \infty} \frac{1}{N} \left(\sum_{k=1}^{N} \xi_k \right)^2$$

and depends on the properties of the autonomous system: the degree of nonisochronism of the oscillations and the statistical characteristics of the amplitude modulation. The parameters Δ and ε are defined by external factors: the detuning of its frequency and amplitude. Note that Eq. (4) is precisely the equation for the phase of a self-excited generator of sinusoidal oscillations which is synchronized by an external periodic signal in the presence of fluctuations [6, 7]. In our case, the inherent stochastic dynamic behavior plays the part of external fluctuaations.

Let us find how an external signal influences the power spectrum. When determining the part of the spectrum that is associated with phase modulation we can ignore amplitude modulation, i.e., we can consider the process

$$x(t)=\sqrt{2}\cos(\psi(t)+\psi_0+\omega_0 t).$$

Here $\omega_0 = 2\pi/T$, ψ_0 is the random initial phase which is uniformly distributed over the interval $[0, 2\pi]$. The spectrum of the process $x(t)$ consists of continuous and discrete components. To obtain the discrete component that appears due to the action of the external signal we have to obtain the asymptotic form of the correlation function for large values of the argument.

We have

$$\begin{aligned} r(\tau) &= \langle x(t)x(t+\tau)\rangle = \\ &= \cos\omega_0\tau\langle\cos[\psi(t+\tau)-\psi(t)]\rangle + \\ &+ \sin\omega_0\tau\langle\sin[\psi(t+\tau)-\psi(t)]\rangle. \end{aligned}$$

Let us take into account the fact that as $\tau \to \infty$ the phases $\psi(t + \tau)$ and $\psi(t)$ can be assumed to be independent random quantities; we then obtain

$$r(\tau)\underset{\tau\to\infty}{\longrightarrow}\cos\omega_0\tau(\langle\cos\psi\rangle^2+\langle\sin\psi\rangle^2).$$

The Fourier transform of this nondecaying part of the correlation function yields the discrete component in the power spectrum at the frequency ω_0 with intensity $s=|\langle\exp(in\psi)\rangle|^2$.

In order to define the quantity s we have to obtain the distribution function of the phase ψ. Let us write the Fokker-Planck equation corresponding to Langevin's equation (4):

$$T\frac{\partial W(\psi,t)}{\partial t}=-\frac{\partial}{\partial\psi}[(\Delta+\varepsilon\cos\psi)W]+\frac{\sigma^2}{2}\frac{\partial^2 W}{\partial\psi^2}. \tag{5}$$

Using the method given in [8], we shall seek the steady-state solution in the form of a Fourier series

$$W(\psi)=\sum_k c_k\exp(ik\psi). \tag{6}$$

Obviously, $s = 2\pi|c_1|$. Substituting (6) into (5), we obtain the set of equations

$$\frac{\sigma^2}{2}ikc_k-\Delta c_k-\frac{\varepsilon}{2}[c_{k+1}+c_{k-1}]=G\delta(k),$$

where the constant G can be obtained from the normalization condition $c_0 = 1/2\pi$.

We shall seek a solution of this set of equations in the form

$$c_{k+1}=-ia_{k+1}c_k,$$

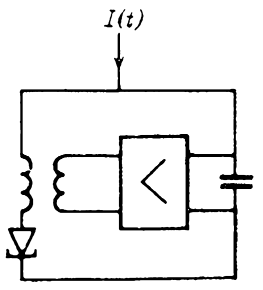
and we then obtain the recurrent relation

$$a_k=\left(a_{k+1}+\frac{2\Delta}{\varepsilon}i+k\frac{\sigma^2}{\varepsilon}\right)^{-1}.$$

The quantity a_1 can then be represented by the infinite continued fraction:

$$a_1=1\Big/\Big(\frac{\sigma^2}{\varepsilon}+\frac{2\Delta}{\varepsilon}i+1\Big/\Big(2\frac{\sigma^2}{\varepsilon}+\frac{2\Delta}{\varepsilon}i+1/\ldots\,.$$

The value of this fraction then gives the desired amplitude of the discrete spectrum, since $|a_1| = 2\pi|c_1| = s$. The fraction was calculated using a computer and the results are shown in



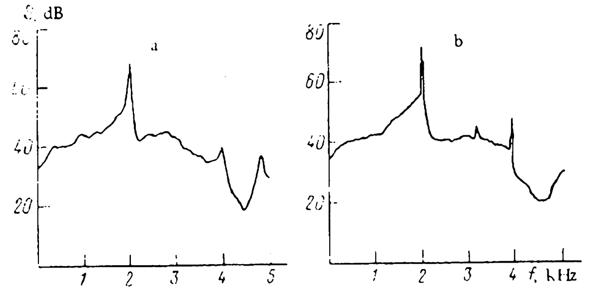


Fig. 3   Fig. 4

Fig. 3. Circuit of the stochastic self-excited oscillator.

Fig. 4. Power spectrum of the oscillations observed in the oscillator shown in Fig. 3; a - I_0 = 0; b - I_0 = 0.167 mA, f_0 = 1980 Hz.

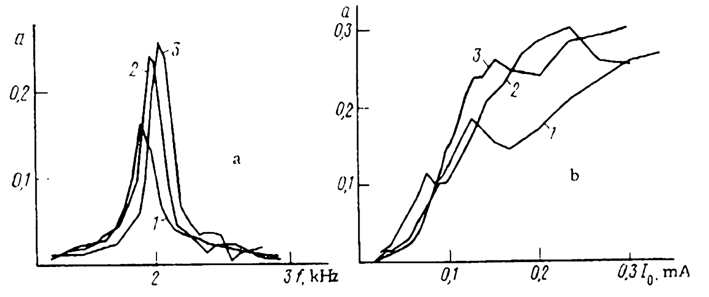


Fig. 5. Amplitude of the discrete component as a function of the external current frequency for I_0 = 0.167 mA (a), and as a function of the amplitude I_0 (b).

Fig. 2. The limits as $\sigma \to 0$ the value of x can be obtained analytically: for $\Delta < \varepsilon$ we have $s = 1$ and for $\Delta > \varepsilon$ we obtain

$$W(\psi) = \frac{\sqrt{\Delta^2-\varepsilon^2}}{2\pi(\Delta+\varepsilon\cos\psi)}$$

$$s = \int W(\psi)\cos\psi\, d\psi = \frac{\Delta}{\varepsilon} - \sqrt{(\Delta/\varepsilon)^2-1}.$$

It is evident in Fig. 2 that the amplitude of the discrete component of s is a maximum for zero detuning. When for a fixed value of Δ we increase the external signal ε, saturation occurs and the amplitude of the discrete component cannot be greater than unity. Synchronization then becomes stronger, the smaller the nonisochronism of the oscillations.

2. EXPERIMENT

The effect of synchronization of the phase of stochastic self-excited oscillations can be experimentally observed using a laboratory setup of a simple noise generator whose circuit is shown in Fig. 3. This generator consists of an oscillatory circuit closed by a positive-feedback amplifier using a tunnel-diode. A detailed investigation of this circuit was discussed in [1]. The spectrum S of the generated self-excited oscillations is shown in Fig. 4a. Here we can see distinctly the pedestal and the peak which correspond to amplitude and phase stochastic modulation, respectively.

The external signal was obtained by introducing a sinusoidal current $I(t) = I_0 \cos \omega_0(t)$. In the region of resonance, when the external signal frequency was within the limits of the peak bandwidth, we observed an increase in the discrete component; the wideband part of the spectrum changed only slightly (Fig. 4b). Figure 5 shows the frequency and amplitude characteristics of the amplitude of the discrete component. Curves 1, 2, and 3 correspond to different degrees of nonisochronism of the oscillations. This parameter varied when the value of the feedback influencing the degree of randomness of the amplitude modulation [1] changed. The mean frequency then shifted somewhat, as is evident in Fig. 5a. The experimental results on the whole confirm the theoretical conclusions.

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