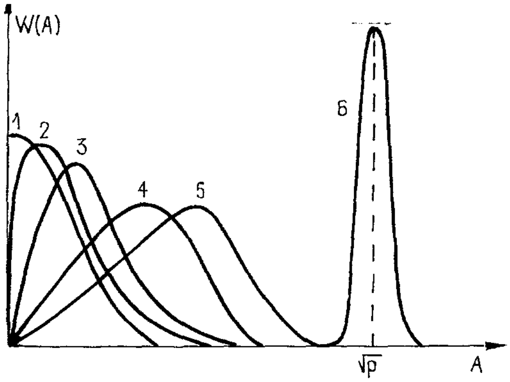

Fig. 1

of transformation of the three-parametric distribution into other types of distribution is depicted in Fig. 1, where curve 1 represents the unilateral normal distribution (16), curve 2 represents the Nakahami distribution (14), curve 3 represents the Rayleigh distribution (15), curves 4 and 5 represent the three-parametric distribution (7), and curve 6 represents the normal distribution (21).

It is obvious, therefore, that the three-parametric distribution generalizes all basic distribution laws which describe amplitude fluctuations in dynamic oscillatory systems affected by wideband noise.

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UNIVERSALITY OF PERIOD DOUBLING BIFURCATION IN ONE-DIMENSIONAL DISSIPATIVE MEDIA

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Universal scaling laws are stated for spatial structures, occurring during transition to chaos in a finite or semiinfinite medium consisting of dissipatively coupled elements. In this case each individual element is a nonlinear system, capable of exhibiting the Feigenbaum sequence of period doubling bifurcation. Results are given of numerical calculations, verifying the laws stated and enabling one to find the functions and constants appearing in the similarity relations.

1. INTRODUCTION

Much attention has recently been devoted to the study of possible transition paths of nonlinear dynamic systems from periodic to chaotic motion [1]. One of the typical scenarios of chaos generation in dissipative systems is related to the existence of a hierarchy of period doubling bifurcations obeying the Feigenbaum scaling law [2]. This scenario is observed in many experiments in hydrodynamics [3], acoustics [4], optics [5], electronics [6,7],

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and chemical kinetics [8]. In the present paper we investigate transition to chaos in an extended medium, being a discrete or continuous set of coupled elements, each of which is individually a nonlinear dissipative system, capable of exhibiting the Feigenbaum sequence of period doubling bifurcations. We provide several examples of these situations.

1) The simplest dynamic system undergoing transition to chaos via period doubling during a parameter change is represented by the one-dimensional recurrent mapping:

$$u_{n+1}=f(u_n,\lambda), \quad \text{e.g.,} \quad u_{n+1}=\lambda(1-2u_n^2). \tag{1}$$

Here n is discrete time and λ is a parameter. The mapping (1) is used, in particular, to describe the dynamics of biological populations [9], in which case the variable u_n has the meaning of population deviation from some level. Considering the spatial population distribution and the dependence of population density on spatial coordinates, we reach a system of the type interesting us.

2) Chemical reaction kinetics is described by nonlinear equations of the form

$$\partial v_i(\mathbf{r},\ t)/\partial t=b_i(v_1, v_2, \ldots, v_k, t)+D_i\nabla^2 v_i, \quad i=1, \ldots, k, \tag{2}$$

where v_i are the concentrations of reacting components, and D_i are diffusion coefficients. In the spatially homogeneous regime relations (2) are transformed to a system of ordinary differential equations, in which transition to chaos via period doubling bifurcation is possible. In the general situation, when the concentrations v_i depend on spatial coordinates, one can mentally partition the volume of the system into small cells, in whose limits the solution is practically uniform. Consequently, we have a set of point systems, capable of exhibiting period doubling and coupled to each other by diffusion.

3) A number of radio-technology and biological systems (see, for example, [10-12]) are described by an equation with delay:

$$\varepsilon dv/dt+v=f(v(t-T),\lambda). \tag{3}$$

Let n and ξ be, respectively, the integral and fractional parts of the number $t/(T + \varepsilon/2)$. Introduce the notation $v(t) = v_n(\xi)$. For $\varepsilon \to 0$ Eq. (3) then reduces to the system of uncoupled mappings $v_{n+1}(\xi) = f(v_n(\xi), \lambda)$. For each fixed ξ we have a mapping of type (1), exhibiting period doubling bifurcation with variation of the parameter λ. Account of the term $\varepsilon dv/dt$ in Eq. (3) can be considered as introduction of coupling between these mappings. The spatial coordinate role is played in this case by the quantity ξ.

4) A distributed Josephson contact in an external periodic field is described by the nonautonomous sine-Gordon equation [13]:

$$\partial^2 v/\partial t^2-\partial^2 v/\partial x^2+\gamma\partial v/\partial t+\sin v=A\cos\omega t. \tag{4}$$

The equation of spatially homogeneous oscillations coincides with the equation for a Josephson point contact, in whose numerical investigation transition to chaos via period doubling has been observed [14]. The distributed contact can be considered as a continuum of interacting point systems. A similar equation is obtained for the problem of crystal dynamics (a lattice consisting of nonlinear oscillators) for a crystal in an intense acoustic or optic field.

The examples provided show that the subject under discussion concerns a wide range of distributed systems, deserving special consideration. The starting point may be the fact that, according to Feigenbaum's theory, uncoupled point systems exhibit universal properties under period doubling. We establish these laws for distributed systems with dissipative coupling between the component elements, i.e., with coupling which tends to balance instantaneous element states. Among the examples given above the systems (1), (2), (3) correspond to dissipative nature of the coupling.

In Sec. 2 we provide a simple model, adequately reproducing the required laws. Keeping in mind that dissipative coupling guarantees stability of the spatially homogeneous regime up to the transition point to chaos, we introduce in Sec. 3 a renormalized group equation for the system evolution operator for a class of states close to homogeneous. Its solution gives the universal factor $\beta = \sqrt{2}$, characterizing the transformation of the spatial state scale during period doubling. Based on this result, in Sec. 4 we formulate scaling hypotheses for the spatial structures generated in subsequent period doubling bifurcations in semiinfinite and confined systems, as well as for the corresponding bifurcation parameter values. These hypotheses are verified by the numerical data given in Sec. 5.

For simplicity we restrict ourselves to the case of a single spatial coordinate, but a similar approach can also be developed for the cases of two or three dimensions.

2. BASIC MODEL

We construct the simplest model of a distributed system for the class under consideration. According to Feigenbaum's theory, near the transition point to chaos each element (cell) of the medium can be described by means of a one-dimensional mapping of type (1), independently of the dimensionality of the phase space of the element. Consequently, the system of uncoupled mappings can be used as a model of a medium of uncoupled elements. The following step introduces the coupling. As to its nature, we make the following assumptions.

1) The coupling between elements is symmetric.

2) The coupling does not affect the dynamics of the spatially homogeneous solution, i.e., the coupling vanishes for equality of instantaneous element states.

3) The coupling is local, i.e., it decreases quite quickly with increasing distance between elements.

4) The coupling has a dissipative nature, implying balance of the instantaneous element states.

To describe the dynamics of the distributed system in this situation we use the following equation:

$$u_{n+1}(x) = \hat{F}[u_n(x)] = \hat{m}f(u_n(x), \lambda). \tag{5}$$

Here x is the spatial coordinate, $f(u, \lambda)$ is the nonlinear function exhibiting in (1) transition to chaos via period doubling, and $\hat{m}$ is a linear operator describing the coupling.

In the case of a discrete variable x the operator $\hat{m}$ can be written in the general case in the form

$$\hat{m}u(x) = \sum_j c_j u(x-j), \tag{6}$$

and for a continuous x — in the form

$$\hat{m}u(x) = \int c(y)u(x-y)\,dy. \tag{7}$$

The coupling properties formulated above impose the following requirements on the operator $\hat{m}$:

1) Symmetry: $c_j = c_{-j}$, $c(y) = c(-y)$.

2) Normalization: $\sum_j c_j = 1$, $\int c(y)dy = 1$.

3) Locality: the sum $\sum_j j^2 c_j = \Delta^2$ or the integral $\int y^2 c(y)dy = \Delta^2$ is finite. The quantity Δ determines the spatial scale related to the operator $\hat{m}$, characteristically the diffusion length in one step of discrete time.

4) Dissipativity: the spectrum $m(k) = e^{-ikx}\hat{m}e^{ikx}$ is less than unity in absolute value.

The spectral shape m(k) in the small wave number region follows uniquely from (1)-(4):

$$m(k) = 1 - (k\Delta)^2/2. \tag{8}$$

As we have stated, Eq. (5) allows one to describe quantitatively the behavior of distributed systems near the transition point to chaos via period doubling bifurcation independently of the specific shape of the dynamic equation of the component elements and the method of introducing coupling between them, it is only necessary that the coupling satisfy the requirements stated above. The usefulness of this approach is indicated by the results of renormalized group analysis, as discussed below. The coupling dissipativity condition is the most nontrivial one. For several systems its satisfaction can be established analytically. Thus, for example, for radio-technological systems with delay, in which the signal successively passes through a noninertial element and a linear inertial link, the equations are naturally written in the form (5) [12]. In particular, Eq. (3) transforms into (5) by trans-

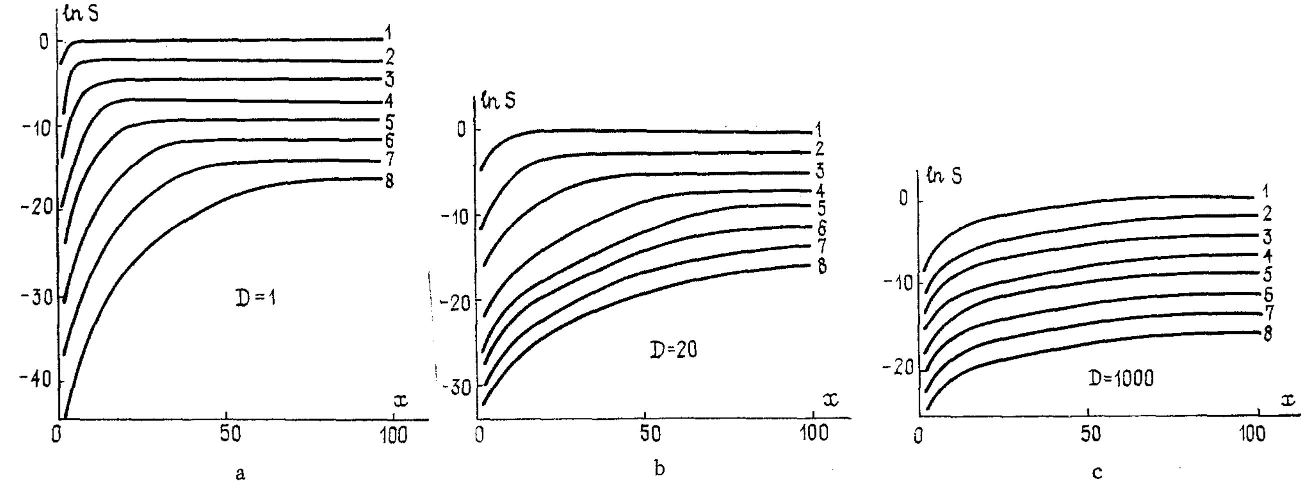

Fig. 1

forming to the variables n, ξ (see above).* Another equation is the equation of chemical kinetics, Eq. (2). As shown in [15], in the case of equal diffusion coefficients $D_1 = D_2 = \ldots = D$ a homogeneous perturbation of a spatially homogeneous solution always increases more quickly (decays more slowly) than the inhomogeneous ones. Thus, the coupling is capable of equalizing states of spatially separated system elements and is dissipative.

3. SCALING RELATION AND ITS CONSEQUENCES

Since dissipative coupling guarantees decay of strongly inhomogeneous perturbations, we consider weakly inhomogeneous states in the system (5). We introduce a scaling relation satisfying the dynamics of these states for subsequent period doubling bifurcations. Following Feigenbaum's idea [2], we seek a renormalized equation for the operator $\hat{F} = \hat{m}f$.

Let there be a weakly inhomogeneous state

$$u(x) = u_0 + \varepsilon u_1(x), \tag{9}$$

where $\varepsilon \ll 1$, and u_0 is independent of x. The action of the operator $\hat{m}f$ on this state leads accurately to ε^2 to the expression

$$\hat{m}f[u(x)] \approx f(u_0) + \varepsilon f'(u_0)\hat{m}u_1(x) \approx f[\hat{m}u(x)]. \tag{10}$$

Two facts follow from Eq. (10): 1) as a result of action of this operator a weakly inhomogeneous state is obtained again; 2) the operations of $\hat{m}$ and f on the given class of states commute. Consequently, double application of the operator $\hat{F}$ gives an operator of the same general form:

$$\hat{F}^2 u(x) = \hat{m}f[(\hat{m}fu(x))] = \hat{m}^2 f^2[u(x)]. \tag{11}$$

We introduce the operation of scale variation of the function u(x):

$$\hat{S}_1 u(x) = \alpha^{-1}u(x), \quad \hat{S}_2 u(x) = u(\beta x), \quad \hat{S} = \hat{S}_1\hat{S}_2, \tag{12}$$

where α and β are constants. We carry out in Eq. (11) the scale transformation $\hat{S}$, and denote the operator obtained as a result by $\hat{F}_1$: $\hat{F}_1 = \hat{S}^{-1}\hat{F}\hat{F}\hat{S}$. The same procedure can be applied to the operator $\hat{F}_1$ etc. As a result we reach the recurrent operator equation:

$$\hat{F}_n = \hat{S}^{-1}\hat{F}_{n-1}\hat{F}_{n-1}\hat{S}. \tag{13}$$

According to (11) we can rewrite $\hat{F}_n$ in the form

$$\hat{F}_n = \hat{M}_n g_n, \tag{14}$$

where

$$\hat{M}_n = \hat{S}_2^{-1}\hat{M}_{n-1}^2\hat{S}_2 = \hat{S}_2^{-n}\hat{m}^N\hat{S}_2^n; \tag{15}$$

$$g_n = \hat{S}_1^{-1} g_{n-1}^2 \hat{S}_1 = \hat{S}_1^{-n} f^N \hat{S}_1^n \tag{16}$$

*The symmetry condition of the operator $\hat{m}$ is satisfied in this case only for small k: $m(k) \sim 1 - (\varepsilon k)^2/2$; this is sufficient, however, for applications of our theory.

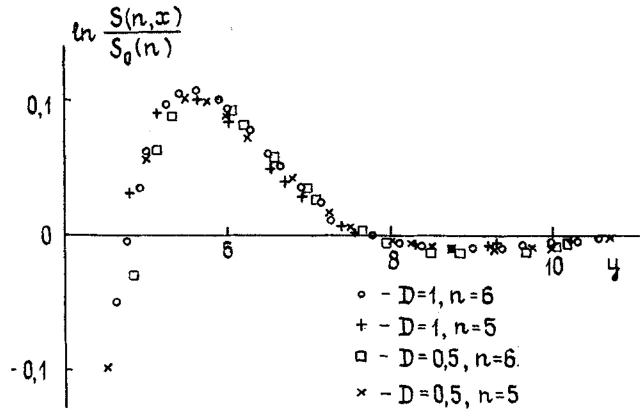

Fig. 2

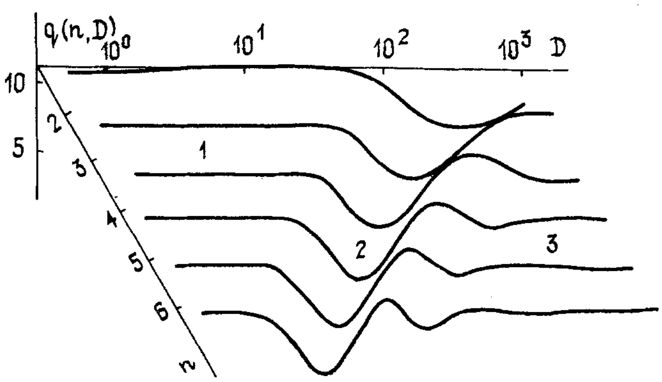

Fig. 3

and the notation $N = 2^n$ was introduced. According to [2], if the parameter λ in $f(u, \lambda)$ equals its critical value λ^0_c, and $\alpha = -2.5029$ is the Feigenbaum constant, then the sequence g_n tends to a universal function. Consider now the limiting behavior of the sequence of operators $\hat{M}_n$. Writing Eq. (15) in spectral representation, we have

$$M_n(k) = [m(k\beta^{-n})]^N. \tag{17}$$

If $\beta > 1$, for $n \to \infty$ the shape of $M_n(k)$ is determined by the behavior of $m(k)$ in the small k region, i.e., by expression (8). Taking the logarithm of (17), we obtain

$$\ln M_n(k) = -(1/2)\Delta^2 k^2 2^n \beta^{-2n}.$$

It is hence clear that the regular limit exists for $n \to \infty$ if one puts $\beta = \sqrt{2}$: $\lim_{n\to\infty} M_n(k) = \exp(-(1/2)\Delta^2 k^2)$.

To sum up, we conclude that the sequence $\hat{F}_n$ converges (for $\lambda = \lambda^0_c$ and the indicated choice of the constants α and β) to the universal operator $\hat{G} = \exp\left(\frac{1}{2}\Delta^2 \frac{\partial^2}{\partial x^2}\right) g$, which is independent of the specific bare operator $\hat{F} = \hat{m}f$ and is a fixed point of the renormalized transformation

$$\hat{G} = \hat{S}^{-1}\hat{G}\hat{G}\hat{S}.$$

It hence follows that for subsequent period doubling bifurcations the system evolution operator during a period remains the same within the choice of the scaling (for sufficiently large n). The characteristic spatial scale related to this operator increases during period doubling by $\beta = \sqrt{2}$ times.

To conclude this section we discuss the consequences of the scaling relation obtained for transition to chaos in an unbounded homogeneous medium. In the point system, let the bifurcation parameter values accumulate to the critical value λ^0_c. The homogeneous state undergoing the same time period doubling bifurcations will be stable in the unbounded medium for $\lambda < \lambda^0_c$. Stochastic oscillations in time and space are generated for $\lambda > \lambda^0_c$. Increasing the time scale by a factor of two corresponds to enhancement of the spatial scale, the correlation radius r_c, by β times. Since doubling of the time scale corresponds to a decrease of $\lambda - \lambda^0_c$ by $\delta = 4.6692$ times [2], we find the critical index for the correlation radius

$$r_c \sim (\lambda - \lambda_c)^{-\nu}, \quad \nu = \log\beta/\log\delta = 0.2249. \tag{18}$$

4. SCALING HYPOTHESES FOR SPATIALLY CONFINED SYSTEMS

4.1. Infinite Medium with a Local Inhomogeneity. The problem of behavior of a medium with a local inhomogeneity is generated, for example, in the following cases: 1) a local perturbation of the parameter λ; 2) a semiinfinite medium $0 \leq x < \infty$ with boundary conditions, say, of the form $u(0) = u_0$; 3) a solution with phase dislocations in an unconfined system. (Under phase dislocation we understand the following. For some λ let the mapping (1) have a

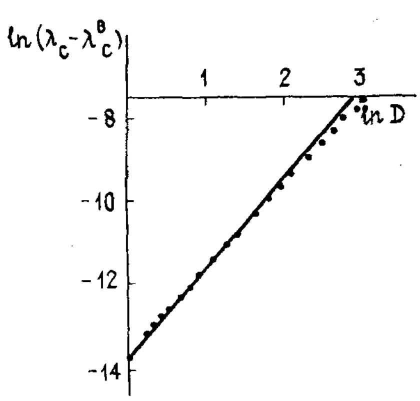

Fig. 4

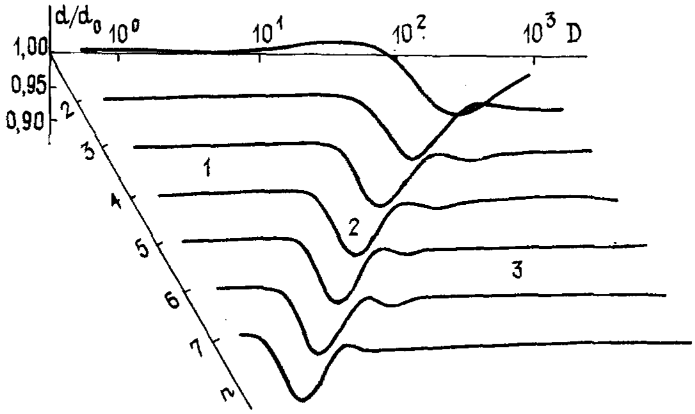

Fig. 5

stable peak of period 2 (u_1, u_2). The system (5) has then a stable solution with a time period 2, such that $u_{2i}(\infty)=u_1$, $u_{2i}(-\infty)=u_2$, $u_{2i+1}(\infty)=u_2$, $u_{2i+1}(-\infty)=u_1$. Though this solution cannot be generated from the stable, spatially homogeneous regime, it can be realized by preparing appropriate initial conditions).

As shown below, the specific type and size of a local inhomogeneity plays no role. The natural condition whose satisfaction is assumed is that the transition to chaos occurs not in the inhomogeneity region, but is determined by the asymptotic solution far from it.

The asymptotic solution is homogeneous for $x \to \infty$. Therefore period doubling occurs at the same parameter values as in a point system. However, the solution generated at the n-th bifurcation with time period 2^n becomes homogeneous only at some characteristic distance ξ_n from the local inhomogeneity. We will call the region in which the solution differs from homogeneous a tail of order n, and the quantity ξ_n – the tail length.

The results of the preceding section make it possible to suggest a tail scaling hypothesis. To simplify its formulation we initially assume that the parameter value equals its critical value $\lambda = \lambda^0{}_c$. We describe the shape of the n-th order tail, characterizing the time period $N = 2^n$, by the quantity

$$S(x,n) = \frac{1}{N}\sum_{i=1}^{N} [u_i(x) - u_{i+N/2}(x)]^2, \tag{19}$$

which equals to the sum of intensities of the spectral components appearing at the n-th bifurcation doubling. According to [16], for the point system the quantity S(n) decreases with increasing n according to the universal law

$$S(n) = \gamma S(n+1), \qquad \ln S(n) = \text{const} - n \ln \gamma, \qquad \gamma \simeq 10.48.$$

In the distributed system, S depends on both n and x. According to the results of the preceding section, it must be assumed that the tail shapes of orders n and n + 1 are similar, and their length are related by $\xi_{n+1} = \beta\xi_n$, i.e.,

$$S(n,x) = \gamma S(n+1, \beta x) \quad \text{or} \quad \ln S(n,x) = \text{const} - n \ln\gamma + \theta(\beta^{-n} x/\Delta). \tag{20}$$

The function θ(y) must be universal, at least for large y. Indeed, in this region the state is weakly inhomogeneous, and the shape of the tail is determined by the universal operator $\hat{G}$.

The scaling relation is generalized to the case $\lambda \neq \lambda^0{}_c$ as follows: if the quantity S(n, x) is calculated for some λ_1, then the right-hand side of Eq. (20) must contain the quantity S(n + 1, βx) for $\lambda = \lambda^0{}_c + (\lambda_1 - \lambda^c_0)\delta^{-1}$. The scaling laws must also be valid behind the critical point. Introducing the quantity s(λ, k), the total intensity of the wideband part of the spectrum, the scaling relation similar to (20) is

$$s(\lambda - \lambda^0 x) = \gamma s((\lambda - \lambda^0_c)/\delta, \beta x). \tag{21}$$

Obviously, the statistically homogeneous chaotic regime is realized far from local inhomogeneity. It can be concluded from Eq. (21) that homogeneity is destroyed at distances of the order of the correlation radius (18).

4.2. A System of Finite Length. Consider now a system of finite length L. For the sake of simplicity we assume that the local inhomogeneity occurs at the left boundary $x = 0$ (for example, $u(0)$ is fixed), while free boundary conditions are given on the right boundary $x = L$. This configuration can be considered as a half-symmetric system of length 2L with inhomogeneities at both boundaries.*

The bifurcation structure in such a system is the following. If the length is large in comparison with the diffusion length Δ, tails are formed for the first bifurcation doublings near the left boundary, while the solution is practically homogeneous near the right boundary. Therefore, the bifurcation points and the spatial configuration of the solution are the same as for a semiinfinite system with a local inhomogeneity. With increasing bifurcation number n the tail length increases proportionately to β^n, so that for some n it is comparable to the system length: $\xi_n \sim L$. A tail interaction with the right boundary is generated in this case, leading to a perturbation of the bifurcation values of the parameter λ_n. According to the scaling hypothesis (20), the tail shape is universal. It can be expected, therefore, that the bifurcation point perturbation is also universal and is determined, roughly speaking, by the tail value at the right boundary. Starting from it, we formulate the scaling hypothesis for the bifurcation parameter values:

$$\lambda_c^0 - \lambda_n = \delta^{-n} K_0 \varphi(\beta^{-n} L/\Delta). \qquad (22)$$

Here λ_c^0 and K_0 are constants in the Feigenbaum equation for a point system ($\lambda_n^0 = \lambda_c^0 - K_0\delta^{-n}$), and ϕ is a universal function. Since in an infinite system bifurcations occur for the same parameter values as in a point system, $\phi(z) \to 1$ for $z \to \infty$.

For further increase in n, it becomes difficult to talk about tails, since their length exceeds the system length. The problem still has a unique spatial scale L, therefore the spatial distribution of spectral components generated again for large n becomes fixed, independent of n. In this region the system is equivalent to its environment, so that doubling bifurcations accumulate to some critical value λ_c according to the usual Feigenbaum law:

$$\lambda_c - \lambda_n = K\delta^{-n}. \qquad (23)$$

Relationship (23) agrees with (22) if it is assumed that for $z \to 0$ $\phi(z) \to -Az^{-\varkappa} + B$, where $\varkappa = 1/\nu = 4.4463$. The constants $\varkappa$, A, B are universal, since the function ϕ is universal. Substituting its expression into (2), we obtain relations of λ_c and K with λ_c^0 and K_0:

$$\lambda_c = \lambda_c^0 + AK_0(L/\Delta)^{-\varkappa}; \qquad (24)$$

$$K = BK_0. \qquad (25)$$

The system behaves as lumped in the supercritical region $\lambda > \lambda_c$ until the correlation radius given by Eq. (18) exceeds the system length. In this case the dynamics is chaotic in time, and the spatial distribution remains fixed. With further increase in the parameter λ, a situation arises in which $r_c < L$. The spatial distribution pattern of motion components becomes then the same as in an unbounded medium with a local inhomogeneity: a region exists far from the supercritical boundary, in which the intensity of noise components falls off.

5. NUMERICAL RESULTS

To verify the scaling hypothesis numerically, we investigated in detail the following discrete system:

$$u_{n+1}(x) = \lambda[1-2u_n^2(x)] + D[u_{n+1}(x-1) - 2u_{n+1}(x) + u_{n+1}(x+1)]. \qquad (26)$$

Equation (26) is a special case of Eq. (5), in which $f(u, \lambda) = \lambda(1 - 2u^2)$, while the Fourier transform of the kernel of the linear operator $\hat{m}$ is

$$m(k) = [1 + 2D(1-\cos k)]^{-1} = 1 - Dk^2 + O(k^4), \qquad (27)$$

so that the diffusion length is $\Delta = \sqrt{2D}$. The system length was $L = 100$, and the boundary conditions were given as follows:

$$u(0) = 0, \quad u(L+1) = u(L). \qquad (28)$$

*In particular, the system with delay (3), described in the Introduction, is reduced to a similar restricted system [12].

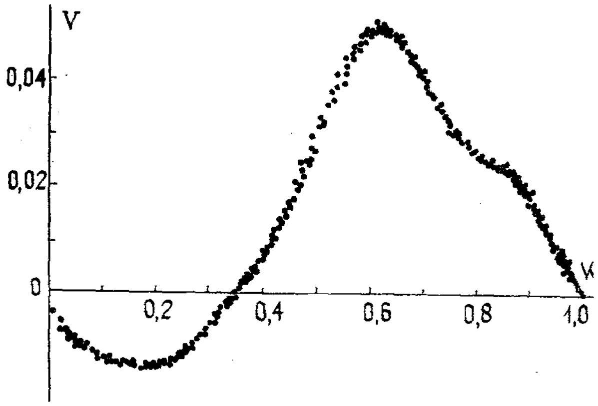

Fig. 6

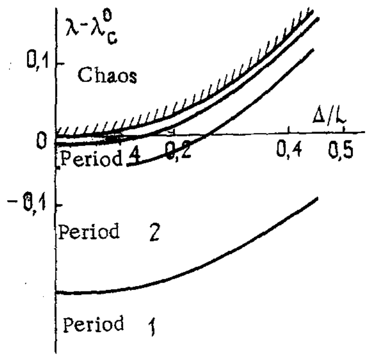

Fig. 7

For various D values within the limits 0.5–1000 bifurcation values were found of the parameter λ for period cycles 1, 2, 4, ..., 256. Also calculated were the quantities $S(n, x)$, determined by Eq. (19), which characterize the intensity components of motion with time period 2^n at the point x.

Figure 1 shows the intensity distribution $S(n, x)$ over the system length for D = 1, 20, 1000. The λ value was selected in each of these cases in such a manner that a stable cycle of period 512 was realized. For D = 1 the increase in the tail length with n is well observed. The tail interaction with the right boundary is small for $n \lesssim 9$, and the tail shape is practically the same as in a semiinfinite system. In the case D = 20 the tail length becomes comparable with the system length for $n \approx 4$. The region n = 4-6 is the transition region to the new regime, when the distribution $S(n, x)$ becomes independent of n. For D = 1000 the distribution $S(n, x)$ does not vary with increasing n already for $n \gtrsim 2$.

Clearly, to verify scaling of tails in a semiinfinite system (Eq. 20) it is necessary to use the calculation results for small D. Figure 2 shows the quantity $\ln[S(n, x)/S_0(n)]$ as a function of the coordinate $y = xD^{-\frac{1}{2}}\beta^{-n}$; here $S_0(n)$ refers to the point system. As seen from Fig. 2, for large n the points corresponding to various n and D approach the same curve, being the graph of the universal function $\theta(y)$. The convergence rate can be enhanced by shifting the point with respect to which the scale is changed, i.e., by using the variable $y = (x + x_*)D^{-\frac{1}{2}}\beta^n$, where x_* is of the order of unity.

Figure 3 shows the dependence on n and D of the intensity ratio of the component of periods 2^n and 2^{n+1} at the right boundary of the system: $q(n, D) = S(n, L)/S(n + 1, L)$. Three regions 1, 2, 3 are clearly seen on the drawing. In region 1 the solution near the boundary is practically homogeneous, and the quantity is near the universal constant γ. Region 2 corresponds to tail interaction with the boundary, and a deviation is observed here between the quantity q and γ. Finally, in region 3 the system behaves as lumped, and q is again near γ.

We turn now to the scaling law for the bifurcation value of the parameter λ. We start with verifying relationship (25). The critical λ_c value was determined for various D by extrapolating the numerically found bifurcation values of the parameter λ. The data obtained are shown graphically on Fig. 4. As could be expected, according to Eq. (25) the points are located along a straight line, whose slope is determined by the constant $\varkappa$. We note that according to Eq. (25) the situation can be improved by replacing L by the quantity $L_{eff} = L + \Delta x_*$, i.e., by taking into account the shift mentioned above of the tail scaling center with respect to the origin of coordinates. Keeping in mind that for the mapping (1) the value of the constant is $K_0 = 0.22$, from the data of Fig. 4 one can determine the universal constant $A \approx 880$.

Some idea on the structure of bifurcations in a confined system is given by Fig. 5, in which is shown the dependence of the ratio $d = (\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n)$ on n and D. Also seen here are the same regions as in Fig. 3: 1) the bifurcation points are the same as in the unbounded system; 2) the tail interaction region with the boundaries; here the deviations of d from the Feigenbaum constant δ are quite large; 3) the region in which the system is equivalent to a lumped one, while the bifurcation points satisfy Eq. (23).

To verify the scaling relation (22) we construct the curve of

$$V = \Phi(W), \tag{29}$$

where

$$W = ((\lambda_c - \lambda_c^0)/(\lambda_c - \lambda_n^0))^{2/\varkappa}, \quad V = (\lambda_c - \lambda_n - \lambda_c^0 + \lambda_n^0)/(\lambda_c - \lambda_n^0),$$

and the quantities marked by the subscript "0" refer to a point system. The function $\Phi(W)$ is related to the universal function $\phi(z)$:

$$\varphi(z) = 1 - Az^{-\varkappa} + (1 + Az^{-\varkappa})\Phi[(1 + z^{\varkappa}A^{-1})^{-2/\varkappa}]. \tag{30}$$

Therefore, the results of calculating the bifurcation values of the parameter λ_n for different D and n, being represented graphically in the coordinates V, W, must lie on a single curve. As seen on Fig. 6, this is indeed the case. Using Eqs. (22), (30), and Fig. 6, one can determine the universal constant B in Eq. (25): $B = 1 - (2/\varkappa)\Phi'(1) \approx 1.08$. Knowing the function $\Phi(W)$, one can easily construct the pattern of regions of the different dynamic regimes on the physical parameter (λ, L)-plane. It is clear from what was said above that this pattern will be universal if one uses the normalized variables $(\lambda - \lambda_c^0)K_0^{-1}$ and Δ/L. The region pattern on the parameter plane is illustrated in Fig. 7. We note that it has properties of scale invariance, i.e., it transforms to itself during a scale variation over $\lambda - \lambda_c^0$ axis by δ times and over the Δ/L axis by $\sqrt{2}$ times.

Besides the models (26) with boundary conditions (28) we have investigated numerically several discrete systems of type (5), differing by the specific choice of the function $f(u,\lambda)$ the operator $\hat{m}$, and the boundary conditions. In particular, we investigated the case of periodic boundary conditions, but with introducing a local inhomogeneity of the parameter λ, as well as a system with boundary conditions $u_{n+1}(0) = u_n(1)$, $u_n(L + 1) = u_n(L)$, which corresponds to phase dislocation. Results were obtained in all these cases, similar to the scaling relation described and verified above and to the universality of the functions $\theta(y)$ and $\phi(z)$.

6. CONCLUSION

We discuss the significance of the results obtained above from the point of view of the general problem of transition to chaos in distributed systems. Firstly, we note the principal aspect following from our results, and having no analog for lumped systems. This is observable in the neighborhood of transition points to chaos in the variation of the spatial configuration of structures generated in the medium after successive bifurcations, and in the existence of definite scaling laws of these structures.

The approach used by us is important methodologically; it makes it possible to construct a distributed medium out of coupled point systems possessing known universal properties. Several different types of universal critical behavior ([2, 17-20] and other references) have been described in the literature. Assigning individual elements to each of these types of universal behavior, one can construct from these elements distributed systems with diverse behavior near the transition point to chaos. We note that in each case one of the possible methods of introducing coupling between elements is dissipative. In this case our approach remains essentially valid. Indeed, it follows from the derivation of relationship (15) that the scale factor β depends on the time transformation factor in the renormalized equation. Let the time scale be enhanced during renormalization by μ times; then $\beta = \sqrt{\mu}$. It hence follows, for example, that if we transform into the chaos region by varying some parameter Λ characterized by the scaling factor δ_Λ, then the spatial scale of structures in a dissipative medium (in the supercritical region — the correlation radius) will vary by the power law $r_c \sim (\Lambda - \Lambda_c)^{-\chi}$, where $\chi = \log \beta/\log \delta_\Lambda$.

In conclusion, we wish to note that our study does not exhaust the problems of transition to chaos in a medium of coupled systems exhibiting period doubling, since dissipative coupling is not exclusively possible. It was shown in [21] that after a large number of period doublings an arbitrary coupling, given by smooth functions of variables determining state elements, reduces to a combination of two universal coupling types. These types possess different transformation properties concerning the renormalized transformation. One of them corresponds to the dissipative coupling considered in the present paper, and leads to the scaling laws described above. Another type allows instability of the homogeneous state of an unbounded system even for $\lambda < \lambda_c$, which substantially complicates the general pattern of transition to chaos.

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NUMERICAL SOLUTION OF THE ONE-DIMENSIONAL PROBLEM OF SELF-INTERACTION OF A WAVE IN A LAYER OF NONLINEAR MEDIUM

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We consider the problem of normal incidence of a plane monochromatic wave on a layer of nonlinear medium with finite thickness whose dielectric permittivity is determined by the intensity of the wave field. We show that the field intensity distribution is invariant with respect to a change of thickness of the layer and the intensity of the incident wave. This invariance can be generalized to three-dimensional problems. For the simplest types of nonlinearity, we study numerically the field at the boundary of the layer and the intensity distribution inside the layer. We investigate the transition to half space.

1. The problem of incidence of a monochromatic wave on a layer of medium $L_0 \leq x \leq L$ whose dielectric permittivity is determined by the intensity of the wave field is of considerable interest, and many works have been devoted to its study (for a bibliography of the problem, see, for example, [1]). In the simplest one-dimensional problem (normal incidence of a wave onto a medium with laminar inhomogeneity) which will be considered here, the wave field in the medium is described by the nonlinear Helmholtz equation

$$(d^2/dx^2)U(x)+k^2[1+\varepsilon(x,J(x))]U(x)=0, \quad J(x)=|U(x)|^2, \tag{1}$$

with the condition of continuity of the field and the derivative at the boundary of the layer. If a wave $U_0(x) = v \exp[ik(L - x)]$ is incident onto the layer, the solution of the problem (1) can be represented in the form $U(x) = vu(x)$, where $u(x)$ satisfies the equation

$$(d^2/dx^2)u(x)+k^2[1+\varepsilon(x,J(x)]u(x)=0, \quad J(x)=w|u|^2, \quad w=|v|^2, \tag{1a}$$

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