Z. Phys. B – Condensed Matter 55, 149–154 (1984)

Condensed Matter
Zeitschrift für Physik B

On the Interaction of Strange Attractors

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Received November 17, 1983

This paper deals with the dynamics of diffusively coupled strange attractors. Such interaction tends to equalize their instantaneous states and, for large coupling constant, results in a homogeneous state that is chaotic in time. The stability of this state depends on the relation between the Lyapunov exponent and the coupling constant. Statistical properties are determined for weakly inhomogeneous disturbances near a stable homogeneous regime. The inhomogeneous state beyond the stability threshold is treated by using the mean-field approximation. We show that both cases of soft (supercritical) and hard (subcritical) excitation of the inhomogeneous state may occur.

I. Introduction

There exist a number of nonlinear dissipative systems that display transitions into a chaotic, or strange attractors (SA) regime [1–3]. This regime has been extensively investigated in the simplest mathematical models (systems of ordinary differential equations (ODE) and discrete mappings) related to different phenomena in fluids, chemical reactions, plasmas, etc. It is advantageous to apply the known properties of the simplest systems to investigate more complex chaotic regimes. One of the possibilities is to consider interaction of strange attractors. As a matter of fact, many systems, including Josephson junctions [4], chemical reactions [5], electronic devices [6] may be regarded as a set (discrete or continuous) of subsystems, each exhibiting chaotic behavior.

In the present paper, we consider diffusive-type interaction between SA, i.e. interaction which tends to equalize instantaneous states of the interacting systems. This type of interaction occurs in chemical reactions, described by reaction-diffusion equations, and in electronic devices coupled with resistors.

Diffusive-type interaction tends to synchronize interacting systems. This is prevented by the exponential instability of the trajectories of the strange attractors. Such instability, quantitatively measured by the Lyapunov characteristic exponent (LCE), is an intrinsic feature of chaotic regimes [7]. Thus one can expect that in the interaction of SA, regimes of two types may be established: *(1)* homogeneous, when all instantaneous states of the subsystems are equal (for a large diffusion/coupling constant) and *(2)* inhomogeneous, when instantaneous states are different (for a small diffusion/coupling constant).

The paper is organized in the following fashion. In Sect. II we develop linear criteria for the stability of a homogeneous state. Three cases are considered here: reaction-diffusion equations, coupled ODE and coupled discrete mappings. In Sect. III we consider weakly inhomogeneous states near a stable homogeneous one, caused by transients or by a small inhomogeneity in the system or environment. Finally, in Sect. IV we present a nonlinear analysis of a weakly inhomogeneous regime beyond the stability threshold. In Sect. III, IV we consider the simplest case of interacting discrete mappings. A brief conclusion is given in Sect. V.

II. Stability of a Homogeneous State

2.1. Reaction-Diffusion Equations

We consider the following system of reaction-diffusion equations

$$\frac{\partial U_i(\mathbf{r},t)}{\partial t}=F_i(U_1,\ldots,U_n,t)+D_i\,\Delta U_i(\mathbf{r},t)$$
$$\mathbf{r}\in G,\qquad i=1,2,\ldots,n \qquad (1)$$

with the zero-flux boundary condition

$$\frac{\partial U_i(\mathbf{r},t)}{\partial l}=0, \qquad \mathbf{r}\in\partial G. \tag{2}$$

Here F_i are the nonlinear functions, D_i are diffusion constants, l is the normal to the boundary of closed domain G.

Equations (1) describe various turbulent-like phenomena in chemistry, biology, etc. [5, 8]. We assume that this system undergoes oscillations homogeneous in space and chaotic in time, i.e. $U_i(\mathbf{r},t)=V_i(t)$ where V_i are governed by the corresponding system of ODE's

$$\frac{dV_i(t)}{dt}=F_i(V_1,\ldots,V_n,t). \tag{3}$$

Spatially homogeneous chaotic oscillations were observed in the Belousov-Zhabotinsky reaction [9-11]. In these experiments the homogeneity of the reaction was maintained by continuous stirring. It is clear, however, that boundary conditions (2) permit a homogeneous regime as an exact solution of (1).

Let us consider the stability of the homogeneous solution. Linearized equations for the disturbances $Q_i(\mathbf{r},t)=U_i(\mathbf{r},t)-V_i(t)$ have the form

$$\frac{\partial Q_i(\mathbf{r},t)}{\partial t}=A_{ij}Q_j(\mathbf{r},t)+D_i\Delta Q_i(\mathbf{r},t) \tag{4}$$

where

$$A_{ij}=\frac{\partial F_i(V_1,\ldots,V_n,t)}{\partial V_j}.$$

Let us expand the solution $(Q_1,\ldots,Q_n)$ in eigenfunctions of the eigenvalue problem:

$$\Delta Y(\mathbf{r})+\nu Y(\mathbf{r})=0, \qquad \mathbf{r}\in G \tag{5}$$

with boundary conditions (2). These eigenfunctions form a complete set, and the eigenvalues are nonnegative: $0=\nu_0<\nu_1\leq\nu_2\leq\ldots$ [12]. Substituting

$$Q_i(\mathbf{r},t)=\sum_k C_{ik}(t)\,Y_k(\mathbf{r}) \tag{6}$$

into (4) we have

$$\frac{dC_{ik}(t)}{dt}=A_{ij}(t)\,C_{jk}(t)-D_i\nu_k C_{ik}(t). \tag{7}$$

In the general case of arbitrarilly different diffusion constants D_i, we cannot solve (7). Thus, in order to treat the problem analytically, we assume all D_i to be equal: $D_1=D_2=\ldots=D_n=D$. Then, changing the variables

$$C_{ik}(t)=B_i(t)\exp(-D\nu_k t) \tag{8}$$

we obtain

$$\frac{dB_i(t)}{dt}=A_{ij}(t)\,B_j(t). \tag{9}$$

Equations (9) exactly coincide with the equations obtained by the linearization of (3) close to the solution $\{V_i(t)\}$. Thus, long-term behavior of the solutions of (9) may be expressed in terms of LCE: $B_i(t)\sim\exp(\lambda t)$ where λ is the maximal LCE [7]. Consequently

$$C_{ik}\sim\exp[(\lambda-D\nu_k)t]. \tag{10}$$

The stability of spatially inhomogeneous disturbances depends on the smallest positive eigenvalue ν_1 (since $Y_0=\text{const}$). Instability occurs when

$$\lambda>\nu_1 D. \tag{11}$$

Otherwise, a spatially homogeneous chaotic regime is stable.

Eigenvalue ν_1 can be determined from (5). If the domain G has the characteristic length L, then $\nu_1\simeq L^{-2}$. Thus, we can introduce the dimensionaless parameter

$$P=\frac{\lambda L^2}{D}. \tag{12}$$

The homogeneous state is stable for $P<P_c$ and unstable for $P>P_c$, where P_c is the critical value of the order of unity.

If the domain G is infinite, then (5) has a continuous spectrum: $Y_\mathbf{k}=\exp(i\mathbf{kr})$. In this case (11) gives the boundary of unstable disturbances: all modes with wavenumbers $k<k_c$ are unstable, where

$$k_c=(\lambda/D)^{1/2}. \tag{13}$$

Thus, the linear stability theory gives the following estimate of the correlation length r_c of spatially inhomogeneous chaotic states in reaction-diffusion systems:

$$r_c\simeq D^{1/2}\lambda^{-1/2}. \tag{14}$$

In the case of different diffusion constants D_i, the above method does not hold. However, one can expect that for close D_i, the stability depends on the parameter P, as before, but D in (12) must be some effective diffusion constant. To demonstrate this point let us consider the following model problem. Chaotic regimes often have the form of a sequence of growing oscillations [9, 13]. Therefore, consider the linear system

$$\frac{\partial U_1}{\partial t} = \alpha_1 U_1 + \beta_1 U_2 + D_1 \Delta U_1$$
$$\frac{\partial U_2}{\partial t} = \beta_2 U_1 + \alpha_2 U_2 + D_2 \Delta U_2. \tag{15}$$

For $\beta_1 \beta_2 - \alpha_1 \alpha_2 < 0$, $\alpha_1 + \alpha_2 > 0$ the homogeneous solution of (15) has the form of exponentially growing oscillations. The growth rate may be regarded as a direct analogy of LCE: $\lambda = 0.5(\alpha_1 + \alpha_2)$. The stability of this solution is readily calculated: mode Y_k is unstable if

$$\lambda = 0.5(\alpha_1 + \alpha_2) > \nu_k (D_1 + D_2)/2 \tag{16}$$

or

$$\nu_k^2 D_1 D_2 - \nu_k(\alpha_1 D_2 + \alpha_2 D_1) + \alpha_1 \alpha_2 - \beta_1 \beta_2 < 0. \tag{17}$$

The condition (16) corresponds to the instability mechanism described above. This may be written in the form (11), if one uses the mean value $D = 0.5(D_1 + D_2)$ as an effective diffusion constant. The condition (17) corresponds to a purely diffusive instability mechanism; this usually leads to dissipative structures [8]. The condition (17) may be satisfied only if D_1 differs considerably from D_2.

2.2. *Ordinary Differential Equations*

Let us consider the interaction of two identical systems described by ODE's:

$$\frac{dX_i}{dt} = F_i(X_1, \ldots, X_n, t) - d(X_i - Y_i)$$
$$\frac{dY_i}{dt} = F_i(Y_1, \ldots, Y_n, t) - d(Y_i - X_i). \tag{18}$$

Equations of this type describe, for example, coupled chemical cells [8] and electronic generators. The stability of the homogeneous chaotic state $X_i(t) = Y_i(t) = V_i(t)$ may be found from the linearized equations for inhomogeneous disturbance $Q_i(t) = X_i(t) - Y_i(t)$ which have a form, analogous to (7):

$$\frac{dQ_i(t)}{dt} = A_{ij}(t) Q_i - 2d Q_i. \tag{19}$$

Thus, disturbance Q_i will damp if

$$\lambda < 2d. \tag{20}$$

The criteria (20) may be easily generalized for the case of three or more interacting systems*.

2.3. *Discrete Mappings*

The simplest system which exhibits chaotic behavior is one-dimensional mapping. The properties of $1-D$ maps have been extensively investigated recently, because they are rather simple and they describe many characteristics of more complex models [1-3].
Let us consider SA in $1-D$ mapping:

$$x_{n+1} = f(x_n, a) \tag{21}$$

where a is the control parameter. Diffusive-type interaction between these SA is described by the following system:

$$x_{n+1} = f(x_n, a_1) + \gamma(y_{n+1} - x_{n+1})$$
$$y_{n+1} = f(y_n, a_2) + \gamma(x_{n+1} - y_{n+1}). \tag{22}$$

Using variables $u = (x+y)/2$, $v = (x-y)/2$ one obtains

$$u_{n+1} = \tfrac{1}{2}[f(u_n + v_n, a_1) + f(u_n - v_n, a_2)] \tag{23a}$$

$$v_{n+1} = \frac{1}{2(1+2\gamma)}[f(u_n + v_n, a_1) - f(u_n - v_n, a_2)]. \tag{23b}$$

Homogeneous chaotic oscillations in identical ($a_1 = a_2$) SAs correspond to the case when $v_n \equiv 0$. The stability of this regime may be determined from the linearized (23b):

$$v_{n+1} = \frac{f'(u_n)}{1+2\gamma} v_n. \tag{24}$$

It follows from (24) that the stability criterion, analogous to (11) and (20), is

$$\langle \ln|f'(u_n)| \rangle = \lambda < \ln|1+2\gamma|. \tag{25}$$

III. Fluctuations Near a Stable Homogeneous State

In the previous section we showed that a homogeneous chaotic regime is stable for sufficiently strong interaction. It is of interest to consider transients to this stable state, as well as weakly inhomogeneous states caused by small inhomogeneous structural disturbances. These problems are not trivial owing to the chaotic nature of a stable state. In this section we shall deal only with the simpleat case of two interacting $1-D$ SA (22).

* After this work was completed we found out that criteria (20), (11) were obtained by Fujisaka and Yamada [16]

3.1. Transients to Homogeneous State

First we rewrite (24) in the following form

$$v_{n+1}=g_n v_n \tag{26}$$

where $g_n=f'(u_n)/(1+2\gamma)$ is the random sequence resulting from the chaotic solution $\{u_n\}$. In new variables $z=\ln|v|$ and $\xi=\ln|g|$ we obtain

$$z_{n+1}=z_n+\xi_n. \tag{27}$$

The Eq. (27) has the solution

$$z_n=\sum_{k=1}^{n-1}\xi_k+z_1. \tag{28}$$

If correlation between ξ_k is small, then applying the central limit theorem, one obtains for $n\gg 1$ the Gaussian probability density $w(z,n)$:

$$w(z,n)=\frac{1}{\sigma\sqrt{2\pi n}}\exp\left[-\frac{(z-na)^2}{2n\sigma^2}\right] \tag{29}$$

where $a=\langle\xi\rangle=\langle\ln|g|\rangle$ and $\sigma^2=\langle\xi^2\rangle-a^2$ are the mean value and the variance of the random variable ξ. The corresponding probability distribution for $|v|$ $W(|v|,n)$ has the form

$$W(|v|,n)=\frac{1}{|v|\,\sigma\sqrt{2\pi n}}\exp\left[-\frac{(\ln|v|-na)^2}{2n\sigma^2}\right]. \tag{30}$$

Using (29) and (30) we can find the evolution of the moments of $|v|$, i.e. $I_m(n)=\langle|v|^m\rangle$:

$$I_m(n)=\exp\left(n\left(am+\frac{m^2\sigma^2}{2}\right)\right). \tag{31}$$

It follows from (31) that during the onset of the homogeneous state, only the moments with $m<-2a\sigma^{-2}$ decrease, while the moments with $m>-2a\sigma^{-2}$ grow exponentially. This may be due to the formation of a power-law tail in the probability density $W(|v|,n)$. Moreover, it is necessary to take into account the fact that the distribution (29) is only approximate. For example, if $\xi_n<0$ for all n, it is clear that $I_m\to 0$ for all m.

If in modelling of (23) by a computer, v_n is smaller than some level (a computer zero), then v_{n+1} and all consecutive $v_{n+2}, v_{n+3}\dots$ are exactly equal to zero. From the point of view of stochastic Eq. (27) this indicates the existence of an absorbing boundary at $z=z_0$.

Let us consider stochastic Eq. (27) with an absorbing boundary, near the stability threshold, i.e. as $|a|\ll 1$. We also assume that the variance σ^2 is small. Then one can approximate the evolution of the probability density (29) by a continuous Fokker-Planck equation

$$\frac{\partial w(z,t)}{\partial t}=-a\frac{\partial w(z,t)}{\partial z}+\frac{\sigma^2}{2}\frac{\partial^2 w(z,t)}{\partial z^2}. \tag{32}$$

According to [14], the corresponding equations for the first two moments of the absorption time of the solution starting at some point z namely $\tau_1(z)=\langle t\rangle$, $\tau_2(z)=\langle t^2\rangle$, are written as

$$\frac{\sigma^2}{2}\frac{d^2\tau_1}{dz^2}+a\frac{d\tau_1}{dz}+1=0 \tag{33}$$

$$\frac{\sigma^2}{2}\frac{d^2\tau_2}{dz^2}+a\frac{d\tau_2}{dz}+2\tau_1(z)=0 \tag{34}$$

with the boundary conditions $\tau_1(z_0)=\tau_2(z_0)=0$. Equations (33) and (34) are readily solved:

$$\tau_1(z)=-\frac{z-z_0}{a} \tag{35}$$

$$\tau_2(z)=-\frac{\sigma^2}{a^3}(z-z_0)+\tau_1^2(z). \tag{36}$$

It follows from (35) and (36) that after finite intervals of time transients to the homogeneous state drop to zero. Near the stability threshold the average duration of the transient process increases proportionally to $|a|^{-1}$ and its variance increases proportionally to $|a|^{-3}$.

3.2. Weakly Inhomogeneous State

We have focused above on the interaction of purely identical strange attractors. If the parameters of interacting systems are not equal (or external inhomogeneous noise is present), the homogeneous state is no longer the exact solution of governing equations. However, one can deal with small inhomogeneous disturbances as a perturbation of the stable homogeneous state, resulting in a weakly inhomogeneous regime.

Let us consider the interaction of two nonidentical SA (22). Linearizing (23) we obtain

$$\begin{aligned} u_{n+1}&=\tfrac{1}{2}[f(u_n,a_1)+f(u_n,a_2)]+\eta_n\\ v_{n+1}&=g_n v_n+\xi_n. \end{aligned} \tag{37}$$

Here

$$g_n=\frac{1}{2(1+2\gamma)}\left[\frac{\partial}{\partial u_n}(f(u_n,a_1)+f(u_n,a_2))\right]$$

$$\xi_n=\frac{1}{2(1+2\gamma)}[f(u_n,a_1)-f(u_n,a_2)]+\zeta_n.$$

In (37) external noise η and ξ are taken into account. Of interest are the statistical characteristics of inhomogeneous component v.

Due to the chaotic nature of the process $\{u_n\}$, variables g and ξ may be regarded as random ones with the probability distributions $R(g)$ and $V(\xi)$, respectively. We assume them to be independent; then the probability density $W(v, n)$ obeys the Chapman-Kolmogorov Eq. [15]

$$W(v, n+1) = \int_{-\infty}^{\infty} K(v, y)\, W(y, n)\, dy \tag{38}$$

where

$$K(v, y) = \frac{1}{|y|} \int_{-\infty}^{\infty} V(v-x)\, R\left(\frac{x}{y}\right) dx.$$

The stationary probability density $W_s(v)$ may be obtained from the equation

$$W_s(v) = \int_{-\infty}^{\infty} K(v, y)\, W_s(y)\, dy. \tag{39}$$

The analytic solution of (39) can be found only for simple cases. Let us assume, for example, that $g_n \equiv g < 1$ (this case corresponds to piecewise linear mapping) and ξ is the Gaussian random variable with the mean ε and the variance ρ^2. It is easy to verify that (39) has a Gaussian solution with the mean $\varepsilon(1-g)^{-1}$ and the variance $\rho^2(1-g^2)^{-1}$. This example demonstrates the increased fluctuations near the stability threshold $g \to 1$.

Another possible way of treating the statistical properties of v is to derive equations for the moments $I_m = \langle v^m \rangle$ directly from (37). After averaging the m-th power of (37) we have

$$I_m = \sum_{k=0}^{m} C_m^k G_k I_k H_{m-k} \tag{40}$$

where

$$G_k = \langle g^k \rangle, \quad H_k = \langle \xi^k \rangle.$$

Using (40) we obtain the following expression for the moments:

$$I_m = \frac{1}{1-G_m} \sum_{k=0}^{m-1} C_m^k G_k I_k H_{m-k}. \tag{41}$$

It follows from (41) that the moment I_m is finite only if $G_l < 1$ for $l = 1, 2, \ldots, m$. Thus, as the control parameter approaches the stability threshold some moments may become infinite before the instability of the homogeneous state is reached. This agrees with the power-law asymptotics of probability density derived above (see (30), (31)).

IV. The Nonlinear Regime above the Stability Threshold

In order to consider the weakly nonlinear regime beyond the stability threshold, we must retain the lowest-order nonlinear terms in (23). Then, assuming again that $a_1 = a_2$ we obtain

$$u_{n+1} = f(u_n) + \tfrac{1}{2} f''(u_n)\, v_n^2 \tag{42a}$$

$$v_{n+1} = \frac{1}{1+2\gamma} [f'(u_n)\, v_n + \tfrac{1}{6} f'''(u_n)\, v_n^3]. \tag{42b}$$

Of interest is the stationary state with small v. Taking the logarithm of the modulus of (42b) and averaging it we have

$$\langle \ln |f'(u_n) + \tfrac{1}{6} f'''(u_n)\, v_n^2| \rangle = \ln |1+2\gamma|. \tag{43}$$

For small v the left-hand side of (43) can be expanded with respect to v:

$$\langle \ln |f'(u_n)| \rangle + \tfrac{1}{6} \left\langle \frac{f'''(u_n)}{f'(u_n)}\, v_n^2 \right\rangle = \ln |1+2\gamma|. \tag{44}$$

In order to treat the problem analytically we use hereafter the mean-field approximation, i.e. we put $v_n^2 = c = \text{const}$. The (44) may be rewritten as

$$\langle \ln |f'(u_n)| \rangle + \tfrac{1}{6} \left\langle \frac{f'''(u_n)}{f'(u_n)} \right\rangle c = \ln |1+2\gamma|. \tag{45}$$

In (45) we cannot replace $\langle \ln |f'(u_n)| \rangle$ by LCE because there is feedback influence on the process $\{u_n\}$. Equation (42a) in the mean-field approximation takes the form

$$u_{n+1} = f(u_n) + \tfrac{1}{2} f''(u_n)\, c = h(u_n, c). \tag{46}$$

LCE for the mapping (46) depends on c as a parameter:

$$\lambda(c) = \left\langle \ln \left| \frac{\partial h(u_n, c)}{\partial u_n} \right| \right\rangle \simeq \langle \ln |f'(u_n)| \rangle + \tfrac{1}{2} \left\langle \frac{f'''(u_n)}{f'(u_n)} \right\rangle c. \tag{47}$$

For small c, $\lambda(c) \simeq \lambda(0) + c\lambda'(0)$ and we obtain the following expression for $\langle \ln |f'(u_n)| \rangle$:

$$\langle \ln |f'(u_n)| \rangle = \lambda(0) - \lambda'(0)\, c - \tfrac{1}{2} \left\langle \frac{f'''(u_n)}{f'(u_n)} \right\rangle c. \tag{48}$$

Substituting (48) into (45) yields the final expression for c:

$$c = \langle v^2 \rangle = 3[\lambda(0) - \ln |1+2\gamma|] \left[\left\langle \frac{f'''(u_n)}{f'(u_n)} \right\rangle - 3\lambda'(0) \right]^{-1} \tag{49}$$

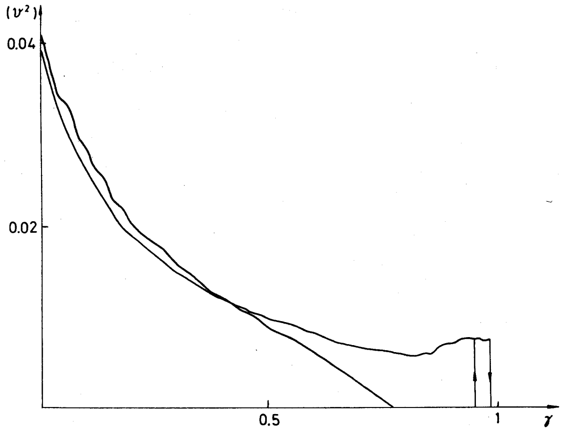

Fig. 1. The dependence of $\langle v^2 \rangle$ on γ for the interaction of mappings (50) with $K=\exp(1)$. Curve 1: $b=-0.2$; curve 2: $b=0.2$

It follows from (49) that the inhomogeneous state may develop in two different ways:
1. Supercritical (soft) transition $1/3\langle f'''(u_n)/f'(u_n)\rangle - \lambda'(0)>0$. In this case a steady-state solution for c exists in the supercritical region ($\lambda(0)>\lambda_c=\ln|1+2\gamma|$) and the intensity of inhomogeneity increases smoothly with the parameter $\langle v^2 \rangle \sim \lambda - \lambda_c$.
2. Subcritical (hard) transition $1/3\langle f'''(u_n)/f'(u_n)\rangle - \lambda'(0)<0$. In this case the steady-state solution for c exists in the subcritical region and is unstable. Thus, a relatively strong inhomogeneous regime is abruptly established with parameter variation.
In real systems both cases may occur. Let us consider, for example, the mapping

$$x_{n+1}=f(x_n)=Kx_n+bx_n^3 \pmod 1 \tag{50}$$

where $K \gg 1$, $|b| \ll 1$.
Application of the above theory yields $h(u,c)=(K+3bc)u+bu^3$, $d\lambda/dc=3bd\lambda/dK$, $f'''=6b$, $f'=K+3bu^2$. Using the smallness of b, we may let $\lambda \simeq \ln K$, $\langle f'''(u_n)/f'(u_n)\rangle \simeq 6bK^{-1}$. Thus, according to (49)

$$\langle v^2 \rangle = -Kb^{-1}(\ln|K| - \ln|1+2\gamma|).$$

Hence, a supercritical (subcritical) transition occurs for a negative (positive) b. This is confirmed by the results of numerical modelling (see Fig. 1). In the subcritical case, a hysteresis was obtained.

V. Conclusion

The diffusive-type interaction between strange attractors which has been discussed here provides a simple model of chaos in complex multi-component systems. However, in many physical situations, interaction of the "inertial" type takes place. Such interaction does not tend to equalize the states of the interacting systems, but results in their oscillations. In the future we hope to extend the above theory to inertial interaction.

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Note Added in Proof

Some properties of the interacting strange attractors were discussed recently by H. Fujisaka and T. Yamada (Prog. Theor. Phys. **70**, 1264 (1983); ibid., p. 1240).