Reprint from
Thermodynamics and Regulation of Biological Processes
Editors: I. Lamprecht, A. I. Zotin

Stochastic Dissipative Structures

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In recent years researchers have shown an increasing interest in those processes proceeding in non-linear non-equilibrium systems and media. This happens because these are wide-spread systems (they are encountered in many fields of physics, chemistry, biology, etc.) and the phenomena observed in them are diversified and beautiful. The analysis of such phenomena is based on simplified models described, as a rule, by differential equations in ordinary or partial derivatives. An important role in these models is played by non-linear equations with diffusion

$$\frac{\partial u(\vec{r},t)}{\partial t} = f_i(u_1,\ldots,u_n) + D_i \Delta u_i(\vec{r},t), \quad i=1,\ldots,n \tag{1}$$

where D_i are diffusion coefficients.

Equations of the type (1) describe distributed chemical reactions, electronic devices, combustion processes, neuron structures, population of organisms, etc. (Vasiliev et al., 1979; Zeldovich, Malomed, 1982). In systems with diffusion, different types of behaviour can be observed, namely, stationary homogeneous regimes, oscillations periodic in space and (or) in time, propagating excitation waves, and, finally, irregular stochastic regimes, both homogeneous and inhomogeneous. The present paper is aimed at the analysis of some types of stochastic behaviour in systems of type (1).

New concepts as to the nature of stochastic processes in non-equilibrium, dissipative systems with aggregated parameters have appeared and been formulated recently. They are associated with the discovery of strange attractors - nontrivial attracting sets in the phase space, on which the solutions to the equations behave in such a tangled and confuse manner that they cannot be distinguished from a random signal, whatever meaning we impart to this term. Such stochastic processes appear in systems without noise and fluctuation and are determined by an exponential instability of individual solutions to small disturbances of the initial conditions (Rabinovich, 1978).

At present most detailed consideration has been paid to chaotic oscillations in the systems described by ordinary, low-order, differential equations as well as to those described by difference equations. There exist descriptions, for example, of interrupted spatially homogeneous regimes in chemical reactions (Turner et al., 1981), ecological populations, etc.. At the same time, spatially inhomogeneous chaotic regimes are also possible due to the distributed character of such systems. In this paper we describe three types of spatially, inhomogeneous, chaotic regimes in systems with diffusion (1). The first type are stationary in time, chaotic, dissipative structures, the second type are slightly spatially inhomogeneous regimes associated with stochastic time dependence, and, finally, the third type are auto-wave order-disorder transition.

Stationary dissipative structures

It is known that in system (1) the stationary homogeneous state, $_i$ = const, can be unstable (if the diffusion coefficients D_i are different), as a result of which there form spatially inhomogeneous structures (1). Stationary structures are described by equations following from (1) if we put $\partial u_i/\partial t = 0$:

$$\frac{d^2 u_i}{dx^2} = -f_i(u_1, \ldots, u_n) \tag{2}$$

System (2) may have chaotic solutions, which corresponds to stationary-in-time and stochastic-in-space structures. We shall show this through an example of the following system

$$\begin{aligned} \frac{\partial u}{\partial t} &= \beta u - \gamma v + D_1 u_{xx} \\ \frac{\partial v}{\partial t} &= -\nu v + g(u) + D_2 v_{xx} \,. \end{aligned} \tag{3}$$

These equations describe the one-dimensional version of the known predator (v) - prey (u) model, where a predominant factor limiting the population number is a non-linear sating of the predator.

In the stationary case ($\partial/\partial t \equiv 0$) from (3) there follows a fourth-order equation

$$D_1 D_2 u^{IV} + (D_2\beta - D_1\nu)u'' - \nu\beta u + \gamma g(u) = 0. \quad (4)$$

The stationary, spatially homogeneous state, $u = u_0$, $v = \beta u_0/\gamma$ is determined from the relation

$$g(u_0) = \gamma^{-1}\nu\beta u_0. \quad (5)$$

By expanding g in a series and restricting ourselves to the quadratic term, we obtain for small deviations from the homogeneous state, $y = u - u_0$,

$$D_1 D_2 y^{IV} + (D_2\beta - D_1\nu)y'' + (\gamma g'(u_0) - \nu\beta)y + \tfrac{1}{2}g''(u_0)y^2 = 0 \quad (6)$$

It is convenient to transform to dimensionless variables

$$\begin{aligned} y &= [2(\gamma g'(u_0) - \nu\beta)/\gamma g''(u_0)]\,\omega \\ x &= (D_1 D_2/(\gamma g'(0) - \nu\beta)^{1/4})\,\xi \end{aligned} \quad (7)$$

and then (6) is reduced to the form

$$\omega^{IV} + \varepsilon\omega'' + \omega - \omega^2 = 0 \quad (8)$$

where the dimensionless parameter ε is equal to

$$\varepsilon = (D_2\beta - D_1\nu)/(D_1 D_2(\gamma g'(u_0) - \nu\beta))^{1/2} \quad (9)$$

(8), considered as an equation of motion in four-dimensional phase space, ($\omega, \omega', \omega'', \omega'''$) has the integral

$$E = 6\omega'\omega''' - 3(\omega'')^2 + 3\varepsilon\omega'^2 + 3\omega^2 - 2\omega^3 = \text{const} \quad (10)$$

Accordingly, the solutions (8) lie on the three-dimensional surfaces E E = const. Real values of the integral E are determined by the boundary conditions. Let us assume that the structures appear in an unlimited region

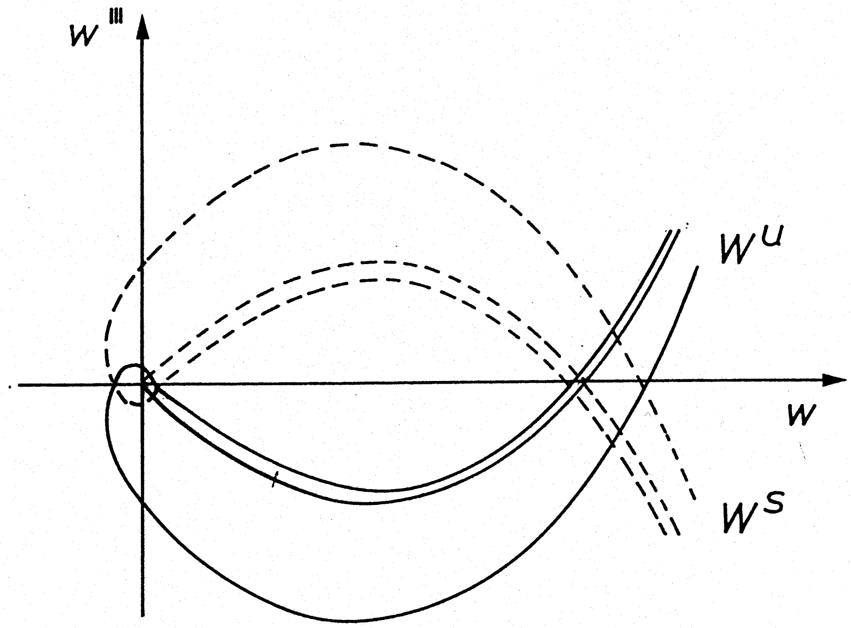

Fig. 40 Explanation in the text.

$-\infty < \xi < +\infty$, and be localised, i. e. $\omega', \omega'', \omega''', \omega \to 0, \xi \to \pm\infty$.
For such structures E = 0.

Localised structures in phase space (8) are described by homoclinic trajectories which tend to the origin at $\xi \to +\infty$ and at $\varepsilon \to -\infty$. Homoclinic trajectories are lines of intersection of an unstable, (W^u), and stable, (W^s), manifold of zero equilibrium, at $|\varepsilon| < 2$. These manifolds are two-dimensional.

These trajectories cannot be determined analytically, and thus the problem has been solved through computation. The cross-sections W^u and W^s of the surface $\omega' = 0$, $\omega'' = -(\omega^2 - 2\omega^3/3)^{1/2}$ at $\varepsilon = 1$ are presented in Fig. 40. W^u and W^s are seen to be intersected transversally (at a non-zero angle), and therefore the theorem (Devaney, 1976) of the existence of a countable set of homoclinic trajectories, is applicable. All possible types of localised structures corresponding to these trajectories can be described using the methods of symbolic dynamics (Bowen, 1979). Some solutions are presented in Fig. 41. These solutions can be coupled, in an arbitrary manner, which fact is responsible for the stochastic character of spatially inhomogeneous regimes. From the elementary structures shown in Fig. 41 one can construct periodic solutions, the period being arbitrarily large, and the internal arrangement arbitrarily complicated.

Although the above stochastic structures occur in a dissipative system, (6) describing these structures is conservative. Hence, stochastic, dissipative structures are analogous to stationary solutions of the corresponding conservative equations. In particular, to equation (6) there is associated the problem of stationary waves via a modified equation (Gorshkov et al., 1979). Similar stationary solutions also exist in the problem of spatial structures in a chain of particles with non-linear interaction. A discrete mapping is here an analog of (6).

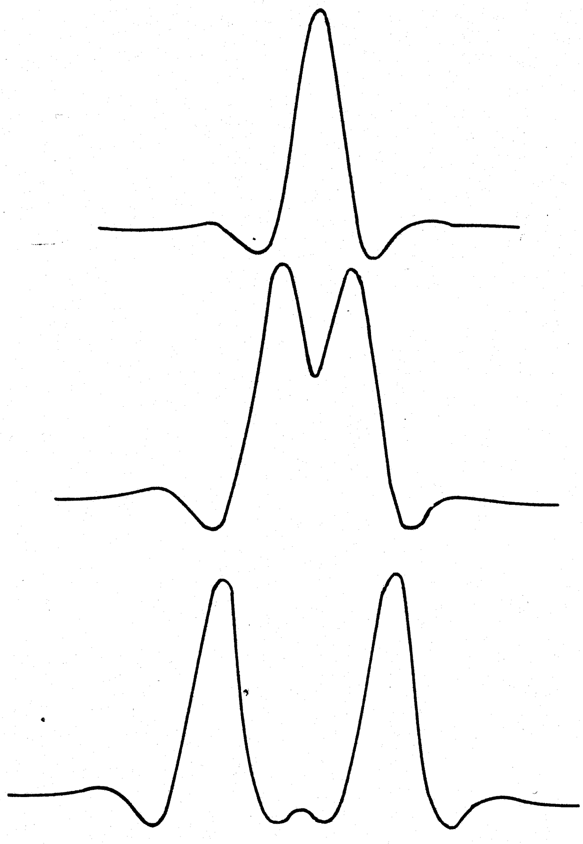

Fig. 41 Explanation in the text.

The appearance of inhomogeneous regimes of stochastic oscillations

Let us consider the system described by (1) with the boundary condition of the absence of flux:

$$\frac{\partial u_i(\vec{r},t)}{\partial \vec{\ell}} = 0 , \qquad \vec{r} \in \partial G \tag{12}$$

where $\vec{\ell}$ is the vector of the normal to the surface of a closed region G. We assume, that in this system, there exist spatially homogeneous stochastic oscillations $u_i(\vec{r},t) = v_i(t)$, where v_i satisfy the system or ordinary differential equation

$$\frac{\partial v_i}{\partial t} \quad f_i(v_1,\ldots,v_n,t) \qquad i=1,2,\ldots,n \tag{13}$$

Let us consider the dynamics of spatially-inhomogeneous perturbations $w_i(\vec{r},t)$ of this solution. On linearising (1), we obtain

$$\frac{\partial w_i}{\partial t} = A_{ij}(t) w_i(\vec{r},t) + D_i \Delta w_i(\vec{r},t) \tag{14}$$

where $A_{ij} = \partial f_i(v_1,\ldots,v_n,t)/\partial v_j$.

Let us expand the solution ($\omega_1,\ldots,\omega_n$) in eigenfunctions of the auxiliary problem

$$\Delta y(\vec{r}) + \nu y(\vec{r}) = 0 \qquad \vec{r} \in G \tag{15}$$

with the boundary conditions (12). As is known, the eigenfunctions (15) $\{y_k\}$ form a whole set, and the eigenvalues are positive $0 = \nu_0 < \nu_1 \leq \nu_2 < \ldots$, (Tricomi, 1954). Writing the expansion in the form

$$w_i(\vec{r},t) = \sum_k W_{ik}(t) y_k(\vec{r}) \tag{16}$$

and substituting into (14), we have

$$\frac{dW_{ik}(t)}{dt} = A_{ij}(t)W_{jk}(t) - D_i \nu_k W_{ik}(t). \tag{17}$$

In the general case of arbitrary coefficients, D_i, it is impossible to analyse the solutions (17). Thus we first consider equal diffusion coefficients $D_I = D_2 = \ldots = D_n = D$. Then, making in (17) the substitution of variables

$$W_{ik} = \exp(-D\nu_k t) Z_i(t) \tag{18}$$

we obtain for

$$dZ_i/dt = A_{ij}(t) Z_j . \tag{19}$$

System (19) exactly coincides with the equations derived in linearising (13) near the solution $\{ v(t) \}$. As is known, the criterion of stochastic oscillation is an exponential increase in disturbances (Rabinovich, 1978), a and at long times the solutions (19) behave like exp (λ t), where λ is the Lyapunov characteristic exponent (Ruelle, 1979). Accordingly,

$$W_{ik} \sim \exp[(\lambda - D\nu_k)t] .$$

Thus, a disturbance corresponding to $\nu_0 = 0$ always increases, but this does not lead to spatially inhomogeneous regimes since ψ_0 = const. Stability toward spatially inhomogeneous disturbances is determined by the eigenvalue ν_1 . Instability occurs if

$$\lambda > \nu_1 D . \tag{20}$$

Otherwise, the regime of spatially homogeneous stochastic oscillations is stable.

In the case of different diffusion coefficients D_i this method does not hold, but one can expect, however, that if the diffusion coefficients differ only a little, the stability is determined, as before, by criterion (20), where D is some mean diffusion coefficient. Since stochastic oscillations often have the form of a succession of groups of increasing oscillations (Devaney, 1976; Rabinovich, 1978), we consider as a model the linear system

$$\dot{u}_1 = \alpha_1 u_1 + \beta_1 u_2 + D_1 \Delta u_1 \tag{21}$$

$$\dot{u}_2 = \beta_2 u_1 + \alpha_2 u_2 + D_2 \Delta u_2 \tag{22}$$

At $\beta_1\beta_2 - \alpha_1\alpha_2 < 0,\ \alpha_1 + \alpha_2 > 0$ (22) has a uniform solution in the form of oscillations with an exponentially increasing amplitude. An analog of the Lyapunov characteristic exponent is here, the increment $\lambda = 0.5(\alpha_1 + \alpha_2)$. Investigation of stability yields the following result: a perturbation corresponding to the mode, y_k, increases if,

$$\lambda > \nu_k (D_1 + D_2)/2 \tag{23}$$

or

$$D_1 D_2 \nu_k^2 - \nu_k(\alpha_1 D_2 + \alpha_2 D_1) + \alpha_1\alpha_2 - \beta_1\beta_2 < 0. \tag{24}$$

Condition (23) coincides with (20) if, for the effective coefficient D, one takes arithmetic mean. Condition (24) corresponds to a purely diffusion instability mechanism. To fulfill this condition, the diffusion coefficients must differ sufficiently.

'Order-disorder' transformation waves

The presence of diffusion in a non-equilibrium medium is known to cause various instabilities whose development may result in the establishment of stochastic regimes. Among the different possibilities for establishment of such regimes, we consider, here, stationary auto-waves which describe transition from a static (unexcited) state of the medium to a regime of stochastic space-time pulsations, and also, transformation waves to which there corresponds a boundary, moving at a constant velocity, and separating in space, the regions of regular stochastic oscillations. As a basic model we consider a two-component onedimensional medium described by equations of the form (1)

$$\frac{\partial u}{\partial t} = u + u^3 - \alpha u w + D_1 \partial^2 u/\partial x^2 \tag{25}$$
$$\frac{\partial w}{\partial t} = \beta w - \delta w^3 - \gamma u^2 + D_2 \partial w/\partial x^2$$

The stationary transformation waves in question depend only on the moving coordinate $\xi = x - v t$, and they satisfy the equations

$$\frac{d^2 u}{d\xi^2} + \frac{v}{D_1}\frac{du}{d\xi} + \frac{1}{D_1}(1 + u^2 - \alpha w)u = 0 \tag{26}$$
$$\frac{d^2 w}{d\xi^2} + \frac{v}{D_2}\frac{dw}{d\xi} + \frac{1}{D_2}(3w - \gamma u^2 - \delta w^3) = 0$$

These equations describe, simultaneously, a system of two related non-linear oscillators with damping. At $\xi \to 0$ all the solutions (26) tend to zero equilibrium. If v is small, the character of motion is close to stochastic oscillations which exist in (26) at $v = 0$. Thus, (26) describe the stationary 'order-disorder' transition wave.

Concluding remarks

The above examples confirm that in non-equilibrium, distributed systems, dissipative stochastic structures are possible which are in no way connected with random inhomogeneities or non-stationary states of the medium. In one-dimensional media parts of such random structures can be located between periodic structures or can be in agreement with unexcited regions of non-equilibirum medium. The question of spatial evolution of chaotic region in two-dimensional media seems to be very interesting. One of the simplest possibilities here is stochastic, cylindrical or spiral, waves induced by a source of such waves working in a stochastic regime, i. e. by a leading centre or reverberator.

Corresponding point models which refer to chemical kinetics and biology have been thoroughly studied.

We would like to mention here another class of stochastic dissipative structures whose dynamics is determined by an external field. An external field can change local parameters of the medium, for example, the velocity of increase of individual components or the magnitude of coupling between components. These changes manifest themselves in the appearance of stochastic patterns.