21. S. P. Kysnetsov, Pis'ma Zh. Tekh. Fiz., 9, No. 2, 94 (1983).
22. A. I. Khibnik, "Periodic solutions of systems of n differential equations. Fortran algorithms and programs," No. 5 (1979), Pushchino, ONTI, Scientific Center of Biological Research.

SYNCHRONIZATION AND STOCHASTIZATION OF ARRAY OF SELF-EXCITED OSCILLATORS BY EXTERNAL NOISE

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UDC 517.9

The study deals with the behavior of an array of noninteracting self-excited oscillators under the action of external delta-correlated pulse noise. A stochastic representation is constructed which relates the amplitudes and the phases of successive pulses. Depending on the stability of motion, in the phase space there occurs either synchronization with all self-excited oscillators coinciding in phase and amplitude or stochastization with the phases noncorrelated. It is demonstrated that transition from synchronization to stochastization occurs as the noise intensity and the anisochronism of oscillations increase.

A study of self-excited oscillatory systems with many degrees of freedom is of great interest in connection with laser, excitable media, and other research [1]. In the analysis of the effect which external fields (regular or random) have on such systems one often assumes that the external action is weak in comparison with the interaction between individual oscillators (self-oscillation modes) [2]. We will consider here the opposite extreme of weak interaction between oscillators, so weak as to be negligible in the first approximation. Our problem then is the behavior of an array of noninteracting oscillators in an external field. This behavior is determined, first of all, by the dynamics of an individual oscillator and, secondly, by the degree of correlation between the amplitudes and between the phases of representatives of the array. Under a periodic external force, for instance, the simplest Thomson oscillator can behave in two possible modes. In the synchronization mode [1] there occur periodic self-excited oscillations whose phases are determined uniquely by the external force and, therefore, all elements of the array are in phase and their response is periodic. Under a large external force the behavior becomes stochastic [3] with the phase varying over a wide range. In this case, the response of the array will be stochastic with a magnitude much smaller than in the synchronization mode, because of the weak correlation between the phases of individual oscillators.

In this study will be considered analogous effects in an array of self-excited oscillators under the action of an external random field. The response of this array will, obviously, be a random function of time. The nontrivial problem in this case is to determine the degree of correlation between the amplitudes and between the phases of individual oscillators.

The evolution of an array of noninteracting identical oscillators under the action of an external force is most conveniently regarded as the evolution of a set of points corresponding to various initial conditions in the phase space of the dynamic system which describes one oscillator. When the phase volume is contracted in all directions, then the differences between the various initial conditions decrease and the array becomes synchronized: all oscillators stabilize in phase with the same amplitude. When in some directions the distance between neighboring trajectories in the phase space increases, however, then synchronization will not occur and the resultant response of the array will weaken appreciably. As the quantitative indicator characterizing the stability of trajectories in the phase space we select the Lyapunov characteristic exponent, its sign determining whether the self-excited oscilla-

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tions become synchronized or stochasticized. We note here that the dynamics of only one oscillator need be known for calculating the Lyapunov characteristic exponent.

In this study will be established the conditions for synchronization or stochastization of an array of quasiharmonic self-excited oscillators under the action of delta-correlated pulse noise.

1. Derivation of Fundamental Equations. We consider a self-excited oscillatory system describable by the second-order equation

$$\ddot{x} + F(x, \dot{x}) = g(t). \tag{1}$$

We will assume that in system (1) in the autonomous mode ($g \equiv 0$) there occur weakly nonlinear quasiharmonic self-excited oscillations describable by averaged equations for amplitude I and phase ψ [4]:

$$\frac{dI}{dt} = F_1(I), \quad \frac{d\psi}{dt} = F_2(I), \tag{2}$$

$$x = I \sin\psi, \quad \dot{x} = I\omega_1 \cos\psi, \quad \omega_1 = F_2(0).$$

Let the amplitude of stable self-excited oscillations be I_0 [$F_1(I_0) = 0$], and their frequency be $\omega_0 = F_2(I_0)$. In the vicinity of this limit cycle it is permissible to linearize equations (2), as a result of which

$$dI/dt = -\gamma(I - I_0),$$
$$d\psi/dt = \omega_0[1 + \alpha(I - I_0)/I_0], \tag{3}$$

where $\gamma = -F'_1(I_0)$ is the decrement of deviations of the amplitude from its steady-state level and the parameter $\alpha = I_0\omega^{-1}_0F'_2(I_0)$ characterizes the anisochronism of oscillations.

Let $g(t) = \varepsilon\omega_1\delta(t)$ represent a single δ-pulse. It does not change x, but it changes $\dot{x}$ by the amount $\varepsilon\omega_1$. With ε a small quantity, we obtain in the first approximation a change of amplitude by $\varepsilon \cos \psi_-$ and a change of phase by $\varepsilon I^{-1}_- \sin \psi_-$ (I_- and ψ_- denote the amplitude and the phase before the pulse).

We now proceed to examine the effect of a stationary random pulse process

$$g(t) = \sigma I_0\omega_1 \sum_i \xi_i\delta(t - t_i) \tag{4}$$

on the self-excited oscillations. Here ξ_i and t_i are, respectively, the random amplitudes of pulses and instants of time of their appearance, with the coefficient σ characterizing the intensity of the noise. We will henceforth assume that $\langle\xi\rangle = 0$ and $\langle\xi^2\rangle = 1$, where $\langle\sigma\rangle$ denotes averaging the external noise over the array.

The equations which relate the amplitudes and the phases before successive pulses can be easily written, inasmuch as the jump under one pulse is known and the evolution between pulses reduces to the process described by the easily integrable equations (3). As a result we have

$$I_{n+1} = I_0 + \exp(-\gamma T_n)(I_n - I_0 + \sigma I_0\xi_n \cos\psi_n),$$
$$\psi_{n+1} = \psi_n + \omega_0 T_n - \sigma(I_0/I_n)\xi_n \sin\psi_n + \alpha\omega_0\gamma^{-1}(1 - \exp(-\gamma T_n))(I_n - I_0 + \sigma I_0\xi_n \cos\psi_n) (\mathrm{mod}\, 2\pi), \tag{5}$$

where $T_n = t_{n+1} - t_n$. We will further assume that ξ_n and T_n are sequences of independent random quantities. Analogous concepts have been developed elsewhere [3] for a periodic sequence of δ-pulses.

It is convenient now to change to the dimensionless variable $x = I/I_0 - 1$ so that

$$x_{n+1} = \exp(-\gamma T_n)(x_n + \sigma\xi_n \cos\psi_n),$$
$$\psi_{n+1} = \psi_n + \omega_0 T_n - \sigma(1 + x_n)^{-1}\xi_n \sin\psi_n + \alpha\omega_0\gamma^{-1}(1 - \exp(-\gamma T_n))(x_n + \sigma\xi_n \cos\psi_n) (\mathrm{mod}\, 2\pi). \tag{6}$$

In the extreme case of long intervals between pulses one can simplify Eq. (6). When $\gamma T_n \gg 1$ at every n (i.e., the distribution function for the random quantity T_n is zero at $T_n < \gamma^{-1}$), then $x_n \ll 1$ and for the phase $\theta = \psi + \arctan(\gamma/\alpha\omega_0)$ we obtain the independent equation

$$\theta_{n+1} = \theta_n + \omega_0 T_n + K\xi_n \cos\theta_n \,(\mathrm{mod}\, 2\pi), \tag{7}$$

where $K = \sigma(\alpha^2\omega^2{}_0\gamma^{-2} + 1)^{1/2}$.

In the next section (No. 2), we will analyze the dynamics of the system in the approximation of long intervals between pulses in accordance with Eq. (7). Then, in the following section (No. 3) we will consider the general case [Eq. (6)].

2. Dynamics of Self-Excited Oscillations under Pulses Separated by Long Intervals. The dynamics of the phase of self-excited oscillations in the case of long intervals between pulses can be described by the stochastic representation (7). In this case the Lyapunov characteristic exponent λ, equal to the average power exponent characterizing the exponential buildup of phase perturbations, can be defined as

$$\lambda = T^{-1}\overline{\langle \ln|d\theta_{n+1}/d\theta_n|\rangle} = T^{-1}\overline{\langle \ln|1 - K\xi \sin\theta|\rangle}, \tag{8}$$

where $T = \langle T_n \rangle$ is the average interval between pulses and the bar denotes averaging over the random-phase distribution function $w_0(\theta)$. It is to be noted that at a $T_n \gg \gamma^{-1}$ the statistics of the instants of time at which pulses appear enter into definition (8) implicitly only, through $w_0(\theta)$. This relates to the fact that most of the phase lead builds up during the time γ^{-1}.

We will determine $w_0(\theta)$ by considering the equation for the evolution of the distribution function $w(\theta)$, this equation being obtainable from known expressions for the probability density of functions of independent random variables

$$w_{n+1}(\theta) = \hat{P}\int_{-\infty}^{\infty} dz \int_0^{2\pi} d\varphi V(\theta - z) W\left(\frac{z - \varphi}{K\cos\varphi}\right)\frac{w_n(\varphi)}{K\cos\varphi}, \tag{9}$$

where V and W are, respectively, $\omega_0 T_n$ and ξ distribution functions, and

$$\hat{P}f(x) = \begin{cases} \sum_i f(x + 2\pi i), & 0 \leqslant x < 2\pi \\ 0, & x < 0,\ x \geqslant 2\pi \end{cases}$$

It is now convenient to change to the characteristic functions

$$w(x) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} c_m e^{imx}, \quad V(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{V}(u)e^{iuy}du,$$

$$W(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{W}(u)e^{iuy}du.$$

Then relation (9) reduces to

$$c_{n+1}(m) = \sum_{l=-\infty}^{\infty} A(l, m)c_n(l), \tag{10}$$

where

$$A(l, m) = \frac{1}{2\pi}\overline{V}(m)\int_0^{2\pi} \overline{W}(mK\cos y)e^{i(l-m)y}dy.$$

Finding the steady-state solution to Eq. (10) for the general case is not a simple problem. It becomes much simpler when the dispersion of intervals between pulses is large: $\langle(\omega_0 T_n - \langle\omega_0 T_n\rangle)^2\rangle \gg 1$. It is noteworthy that this condition does not contradict the earlier approximation $\gamma T_n \gg 1$. In this case, since $\overline{V}(0) = 1$ and $\overline{V}(m) \ll 1$ for $m \neq 0$, we have $A(l,$

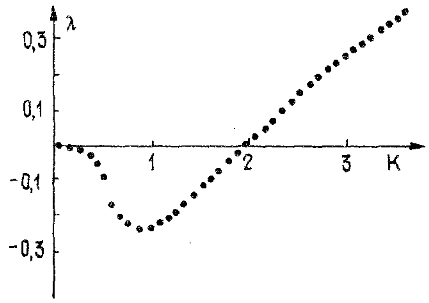

Fig. 1

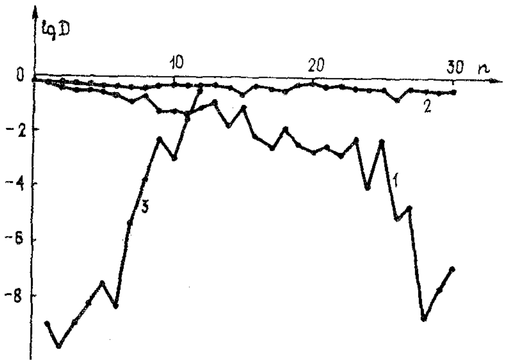

Fig. 2

Fig. 3

m) ≈ δ(m) and the steady-state solution to Eq. (10) is $c_0(m) = \delta(m)$, which corresponds to a uniform phase distribution $w_0(\theta) = 1/2\pi$.

Even with a uniform phase distribution, however, it is not possible to evaluate the integral in expression (8) analytically. In order to determine how the Lyapunov characteristic exponent λ depends on K, we will consider the extreme cases of small and large K.

When K << 1, then

$$T\lambda = \overline{\langle -K\xi \sin\theta - (1/2)K^2\xi^2 \sin^2\theta \rangle} + O(K^3) = -(1/2)K^2\overline{\sin^2\theta} + O(K^3) < 0. \quad (11)$$

When K >> 1, then

$$T\lambda \simeq \overline{\langle \ln|K\xi \sin\theta| \rangle} = \ln|K| + \langle \ln|\xi| \rangle + \overline{\ln|\sin\theta|} > 0. \quad (12)$$

Accordingly, λ is negative and oscillations become synchronized when K is below the critical K_c, while λ is positive and oscillations become stochasticized when K is above the critical K_c.

A numerical evaluation of the Lyapunov characteristic exponent for representation (7) has been made on the basis of normal ε_n and T_n distributions. The graph in Fig. 1 indicates a fair agreement with estimates (11) and (12). The critical value K_c is approximately 2. The synchronization dynamics and the stochastization dynamics were also analyzed numerically. Representation (7) was applied to an array of 500 oscillatory systems with initial phases uniformly distributed over the interval from 0 to 2π. For determining the evolution process, on each step was calculated the dispersion D of the quantity y = cos θ. The graph in Fig. 2, depicting the evolution of D, reveals the difference between stochastization (K = 5, curves 2 and 3) and synchronization (K = 1, curve 1). The phases were found to be correlated already after the action of 30 pulses with K = 1 and D was found to approach 0.5, corresponding to a uniform phase distribution, even after the phases had initially been almost the same (curve 3) with K = 5.

In concluding this section, we will evaluate the final spread of the amplitudes of self-excited oscillations. Squaring the first of equations (6) and then averaging (using a uniform distribution for ψ) yields

$$\langle \overline{x^2} \rangle = \frac{\langle \exp(-\gamma T_n) \rangle \sigma^2}{2(1 - \langle \exp(-\gamma T_n) \rangle)} \simeq \frac{1}{2} \langle \exp(-\gamma T_n) \rangle \sigma^2. \tag{13}$$

3. Mechanism of Stochastization and Synchronization. The physical mechanism involved in synchronization and stochastization of oscillations has the following ingredients. Under the action of an external pulse there occurs a change in the amplitude and the phase of oscillations, whereupon during the subsequent evolution there builds up an additional phase lead because of the amplitude dependence of the frequency. As a result, the phase before the next pulse depends nonlinearly on the phase before the preceding pulse, the degree of nonlinearity being determined by the parameter K proportional to the pulse amplitude and to the anisochronism of oscillations. This nonlinearity leads to a phase equalization when K is small, while the nonlinear phase lead and an intense noise cause a phase decorrelation when K is large.

When the intervals between pulses are not always long, as typically in a Poisson sequence, then one cannot generally replace representation (6) with representation (7). It is obvious, however, that the aforementioned mechanism of nonlinear phase lead buildup remains valid here, too. Short intervals between pulses contribute to a phase equalization, while long intervals contribute to a phase decorrelation. With the T_n and ξ_n distribution functions fixed, an increase of parameters σ and α results in a larger fraction of intervals on which stochastization will occur. There exist critical values of these two parameters below which the phase, on the average, becomes synchronized and above which they become stochasticized. This has been confirmed by numerical simulation (Fig. 3). These calculations were based on a Poisson distribution of T_n with $T = \langle T_n \rangle = 1$ and a normal distribution of ξ_n, with parameters $\gamma = 2$, $\omega_0 = 80$, and $\alpha = 0.5$ fixed but the intensity σ of the external noise varied. In the process were calculated average-in-time values of the quantity D (after the end of the transient period), which had been introduced here earlier. According to curve 1 in Fig. 3, $\sigma_S \simeq 0.1$ is the stochasticity limit. The phases become synchronized when $\sigma < \sigma_S$ and do not when $\sigma > \sigma_S$.

Analogous calculations were also made for an array of nonidentical systems, with the frequencies of individual oscillators uniformly distributed over the $\Delta\omega$ band. The results of these calculations are also shown in Fig. 3 (curve 2 for $\Delta\omega = 0.01$, curve 3 for $\Delta\omega = 0.1$, curve 4 for $\Delta\omega = 1$). A rather strong phase correlation is in evidence here when $\Delta\omega$ is narrow and $\sigma < \sigma_S$. The phase spread can be evaluated from the equation for phase perturbations

$$\Delta\psi_{n+1} = \exp(\lambda T)\,\Delta\psi_n + \Delta\omega T_n. \tag{14}$$

Here the first term on the right-hand side describes synchronization, namely attenuation of phase perturbations with λ in the exponent, and the second term describes the phase lead caused by frequency mismatch. Taking into account the mutual independence of $\Delta\psi_n$ and T_n, we obtain from Eq. (14) the estimate

$$\langle (\overline{\Delta\psi_n})^2 \rangle \simeq \frac{\Delta\omega^2 \langle T_n^2 \rangle}{1 - \exp(-2\lambda T)} \tag{15}$$

for the dispersion of phase perturbations. According to the graph in Fig. 1, the minimum value of λ is approximately 0.25, so that synchronization will be hardly noticeable when $(\Delta\omega)^2 \sim (\langle T^2{}_n \rangle)^{-1}$.

It has been demonstrated in this study that action of external noise on a self-excited oscillatory system produces unique synchronization and stochastization effects. These effects are analogous to the phenomenon of frequency locking and stochastization caused by action of an external periodic force on a self-excited oscillatory system. While these effects are manifested in the dynamics of an individual self-excited oscillator in the latter case, however, in the case studied here they are manifested only in an array of identical (or almost identical) systems in one external random field. It is noteworthy, furthermore, that the described mechanism of synchronization and stochastization operates also when external noise acts on an array of nonlinear oscillators with attenuation. This problem will be analyzed in a future study.

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LITERATURE CITED

1. P. S. Landa, Self-Excited Oscillations in Systems with Finite Number of Degrees [in Russian], Nauka, Moscow (1980).
2. I. M. Klibanova, A. N. Malakhov, and A. A. Mal'tsev, Izv. Vyssh. Uchebn. Zaved., Radiofiz., 14, No. 2, 173 (1971).
3. G. M. Zaslavsky, Phys. Lett., 69A, No. 3, 145 (1978).
4. N. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in the Theory of Nonlinear Oscillations [in Russian], Nauka, Moscow (1974).

SPECTRUM OF CURRENT FLUCTUATIONS IN EMITTER WITH RANDOM NUMBER OF EMISSION CENTERS

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On the basis of a doubly stochastic Poisson process, a model of emission is constructed which accounts simultaneously for the random nature of the time interval between two consecutive instants of electron escape from the cathode and the randomness of the number of emission centers located on the cathode surface. The spectral power density of emission current fluctuations is then calculated in accordance with this model. It is demonstrated that this spectrum contains a range of an inverse-square (ω^{-2}) frequency dependence and a range of "white" noise.

The spectral power density of emission current fluctuations is usually calculated either on the basis of the shot-effect model [1, 2] or on the basis of any one flicker-effect model [3, 4].

The object of this study is to construct a model of emission which will combine both those effects. Its gist is as follows. In the case of shot noise, as is well known, electrons forming the emission current pass through the observation plane at random instants of time t_ν separated by intervals $\theta = t_{\nu+1} - t_\nu$, which have an exponential distribution

In our model the observation plane will coincide with the plane of the emitter. The source of escaping electrons will be regarded emission centers whose number N(t) at any given instant of time is a random quantity (consequence of physicochemical processes occurring on the emitter surface and in the emitter bulk). When all centers emit independently of one another, then the average number of electrons crossing the observation plane per unit time is also a random quantity expressible as

$$\nu(t) = \nu N(t), \tag{1}$$

where ν denotes the intensity of electron emission by one center.

Accordingly, the emission current J(t) in the observation plane can be described as a doubly stochastic process with conditional moments

$$\overline{J(t)} = M\{J(t)\,|\,\nu(t),\quad 0 \leqslant t\} = q_e \nu(t); \tag{2}$$

$$\overline{J(t)\,J(t')} = M\{J(t)\,J(t')\,|\,\nu(t)\} = q_e^2 \nu(t)\,\delta(t' - t) + q_e^2 \nu(t)\,\nu(t'), \tag{3}$$

where $q_e = 1.6 \cdot 10^{-19}$ K, $\delta(t' - t)$ Is the Dirac delta function, and the bar above a symbol denotes averaging over the array of realizations of the random process n(t) (number of elec-

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