

Multistability and autostochasticity in a laser with a delayed-response active medium subjected to periodic loss modulation

I. I. Matorin, A. S. Pikovskii, and Ya. I. Khanin

Institute of Applied Physics, Academy of Sciences of the USSR, Gorki

(Submitted December 8, 1983)

Kvantovaya Elektron. (Moscow) **11**, 2096–2103 (October 1984)

A theory of multistability of a cw single-mode laser with a delayed-response active medium is developed. It is shown that harmonic modulation of the losses (even to a relatively small depth) generates a complex resonance structure in the laser response. Nonlinear resonances at harmonics and subharmonics of the modulation frequency give rise to numerous hysteretic effects. A transition to a dynamic chaos regime under these conditions may result from bifurcation schemes of both soft and hard types. A theory is developed and compared with the results of experimental investigations of multistability and autostochasticity exhibited by solid-state and CO₂ lasers with periodic modulation of the Q factor of the resonator.

1. INTRODUCTION

Grasyuk and Oraevskii established in 1962 that a regime with nonperiodic self-modulation is in principle possible;^{1,2} they did this by solving numerically a semiclassical system of equations describing a lumped model of a single-mode quantum oscillator. The same system of equations was derived and its solutions were investigated in detail by Lorenz, who was concerned with the problem of thermal convection in the atmosphere.³ He established that in the three-dimensional phase space of the investigated system there is a bounded region of attraction between paths which is organized in a more complex manner than stable equilibrium positions and limit cycles; this region has been subsequently called the strange attractor. Such a region contains an infinite but denumerable number of limit cycles and possibly a finite number of unstable equilibrium states. Inside this region a mapping point moves in a nonperiodic manner and this corresponds to a complex random (chaotic) behavior of a dynamic system not associated with fluctuations or any random effects on the system. A sign of stochasticity is an exponentially increasing separation between paths which are initially adjacent. The argument of the exponential function is a Lyapunov characteristic quantity which is a measure of the rate of mixing of paths and is a convenient criterion of the degree of stochasticity of the investigated dynamic process.⁴

Experimental detection of autostochastic regimes in quantum oscillators (lasers) corresponding specifically to the Lorenz attractor has been found to be extremely difficult because of the very unusual nature of the condition which must be satisfied.^{2,5} Nevertheless, the laser is a fairly attractive object for the experimental investigation of complex dynamic processes. Therefore, attempts have been made to find strange attractors of different nature and to identify the conditions for realization of the corresponding laser regimes. Several dynamic models of a laser have been suggested and these are characterized by random pulsations of the radiation or, in other words, by laser turbulence.⁵ Such regimes have been predicted for multimode lasers with^{6,7} and without^{8,9} a saturable absorber.

Several investigations^{10–12} have shown that stochasticity can appear when a laser is subjected to monochromatic

radiation of frequency close to its natural emission frequency. In the adiabatic case (when the polarization and difference between the populations in the active medium follow instantaneously the changes in the field in the resonator), which represents lasers with rapidly relaxing active media (helium–neon laser, dye lasers), stochasticity appears only in the presence of an additional low-frequency modulation of the external radiation¹¹ or of the pumping.¹² In lasers with delayed-response active media (such as luminescent crystals and glasses, semiconductors, and some molecular gases), the difference between the populations reacts to a change in the radiation field only after a finite delay. Consequently, transient processes are not adiabatic, but oscillatory. In view of the existence of a natural frequency of relaxational oscillations, the simplest method of stochastization of radiation is to limit the external effects to just periodic modulation of the Q factor of the laser resonator.¹³ This autostochasticity variant will be considered theoretically below. The corresponding regimes have been observed experimentally in carbon dioxide¹⁴ and YAG:Nd solid-state¹⁵ lasers.

2. EQUATIONS FOR THE INVESTIGATED LASER MODEL

We shall consider a system of rate equations for a laser generalized in the case of low-frequency harmonic modulation of the losses¹⁶:

$$\frac{dM}{dt} = BMN - \frac{M}{T_c} (1 + \beta \cos \omega t), \quad (1a)$$

$$\frac{dN}{dt} = \frac{N_0 - N}{T_1} - BMN. \quad (1b)$$

Here, M is the number of photons in the laser resonator; B is the Einstein coefficient; T_c is the photon lifetime in the resonator; N is the difference between the populations; N_0 is the unsaturated value of this difference; T_1 is the relaxation time of the difference between the populations; β is the modulation depth; ω is the loss modulation frequency.

A phase picture of the system (1) in the absence of modulation ($\beta = 0$) has been studied quite thoroughly.¹⁶ When the self-excitation condition $N_0 > N_{th} = 1/BT_c$ is satisfied, the trivial equilibrium state corresponding to $N = N_0$ and $M = 0$ becomes unstable and the mapping point approaches along a spiral path a stable equilibrium state $N = N_{th}$,

$M = M_{st} = (N_0 B T_C - 1) / B T_1$. The corresponding frequency of the radiation intensity pulsations in the limiting case of small amplitudes is

$$\omega_0 = [(N_0 / N_{th} - 1) / T_1 T_C]^{1/2}, \quad (2)$$

and the decay time of pulsations is $t_d = 2 T_1 N_0 / N_{th}$. In view of the slow decay of relaxational oscillations relatively weak periodic modulation of the losses is sufficient for the appearance of strong pulsations of the output radiation.,^{16,17}

In the subsequent analysis and numerical calculations it will be convenient to write down the laser equations in a form different from the system (1). This can be done following the procedure of Refs. 18 and 19 and introducing a variable $x = \ln(M / M_{st})$ and by renormalization of the other quantities: $\tau = \omega_0 t$; $\Omega = \omega / \omega_0$; $\theta = \Omega \tau$; $z = \omega_0 T_1 (N - N_{th} / N_{th})$; $\varepsilon_1 = 1 / \omega_0 T_1$; $\varepsilon_2 = \varepsilon_1 (N_0 - N_{th}) / N_{th}$; $R = \beta \omega_0 T_1$. In terms of the new notation, the system (1) becomes

$$dx/d\tau = z - R \cos \theta, \quad dz/d\tau = 1 - (1 + \varepsilon_2 z) e^x - \varepsilon_1 z, \quad (3a)$$

$$d\theta/d\tau = \Omega. \quad (3b)$$

The equations in the system (3) describe a certain flux $\dot{x}, \dot{z}, \dot{\theta}$, in a three-dimensional phase space x, z, θ . Since the divergence of the flux is negative,

$$(\partial \dot{x} / \partial x) + (\partial \dot{z} / \partial z) + (\partial \dot{\theta} / \partial \theta) = -(\varepsilon_1 + \varepsilon_2 e^x), \quad (4)$$

any closed cell in this space evolves so that its volume tends to zero. All the paths eventually reach a region of the phase space bounded along x and z and forming a cylinder with its generator parallel to the axis θ . In a system of this kind we can therefore have complex oscillatory processes.

The quantities ε_1 and ε_2 in Eq. (3) are the small parameters of the problem. For example, in the case of a solid-state (neodymium) laser we have $\varepsilon_1 \approx 10^{-2}$ and $\varepsilon_2 \approx 10^{-3} - 10^{-2}$. The parameter R represents the degree of the influence of external agencies on a laser. This degree is small for $R \ll 1$, whereas for $R \gg 1$ we can expect strongly nonlinear effects. In the case of solid-state lasers the latter case corresponds to modulation of the resonator Q factor to a depth of the order of a percent, because $T_1 / T_C \approx 10^4$.

We shall eliminate the variable z by differentiating the first equation in the system (3) with respect to τ ; we shall then use the whole system (3). This gives the following equation for a nonlinear oscillator characterized by low dissipation:

$$\ddot{x} + (\varepsilon_1 + \varepsilon_2 e^x) \dot{x} + (1 + \varepsilon_2 R \cos \Omega \tau) e^x - 1 = R \Omega \sin \Omega \tau - \varepsilon_1 R \cos \Omega \tau, \quad (5)$$

which is excited by an external force and parametrically. We note that if $\varepsilon_2 = 0$, there is no parametric interaction with the oscillator. Equation (5) has the advantage of being clear, whereas the system (3) is more convenient for numerical solutions.

3. MULTISTABILITY AND HYSTERETIC EFFECTS

Some idea of the solutions of Eq. (5) can be obtained by using the smallness of the parameters ε_1 and ε_2 and applying the harmonic balance method. First of all, we shall find the form of the "skeletal" curve representing a geometric locus of extrema of resonance curves and then the dependence of the frequency of natural oscillations of the investigated con-

servative system on their amplitude.²⁶ With this in mind we shall substitute in Eq. (5) a solution $x = a + b_1 \cos \Omega \tau$, which in the case $\varepsilon_1, R \rightarrow 0$ gives the relationship

$$-\Omega^2 b_1 \cos \Omega \tau + \exp(a + b_1 \cos \Omega \tau) = 1. \quad (6)$$

Replacing $\exp(b_1 \cos \Omega \tau)$ by a power series, we can rewrite Eq. (6) in the form

$$e^{-a} (1 + \Omega^2 b_1 \cos \Omega \tau) = 1 + \sum_{k=1}^{\infty} \frac{b_1^k \cos^k \Omega \tau}{k!}.$$

Equating the constant components on the right- and left-sides of this equation we obtain

$$e^{-a} = 1 + \sum_{k=1}^{\infty} b_1^{2k} / 2^k (k!)^2. \quad (7)$$

Application of the same procedure to the coefficient in front of $b_1 \cos \Omega \tau$ gives

$$\Omega^2 e^{-a} = 1 + \sum_{k=2}^{\infty} b_1^{2k-2} / 2^{2k-2} (k-1)! k!. \quad (8)$$

Substitution of Eq. (7) in Eq. (8) yields the expression

$$\Omega^2 = \frac{\left[1 + \sum_{k=2}^{\infty} b_1^{2k-2} / 2^{2k-2} (k-1)! k! \right]}{\left[1 + \sum_{k=1}^{\infty} b_1^{2k} / 2^k (k!)^2 \right]}, \quad (9)$$

which defines implicitly the skeletal curve $b_1(\Omega)$.

In the limit, $b_1 \rightarrow \infty$, we have $\Omega \rightarrow 0$. This means that for a given type of nonlinearity the skeletal curve and, consequently, the resonance curves of an oscillator slope in the direction of low frequencies, as shown in Fig. 1. The shape of the resonance curves demonstrates the existence of two stable solutions (bistability)¹¹ and hysteretic behavior of the system when the modulation frequency and depth are varied. Two branches of stable solutions exist in the range $\Omega < 1$. These branches differ considerably in respect of the contrast of the solution, i.e., in respect of the ratio of the maximum and minimum values of the radiation intensity at the modulation period, or which is equivalent, the value of the difference $\delta x = X_{\max} - x_{\min}$ (Ref. 20).

In view of the fact that Eq. (5) is strongly nonlinear, the system under investigation may exhibit also higher resonances at harmonics and subharmonics of the external force.²¹ Each of them can be represented by a resonance curve of the type shown in Fig. 1. A general amplitude-frequency characteristic (transverse function) of a laser can be

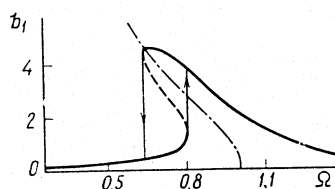


FIG. 1. Dependence of the amplitude of the first harmonic of the solution b_1 on the modulation frequency Ω . The dashed curve is the unstable branch and the chain curve is the skeletal curve. The hysteretic region is identified by arrows.

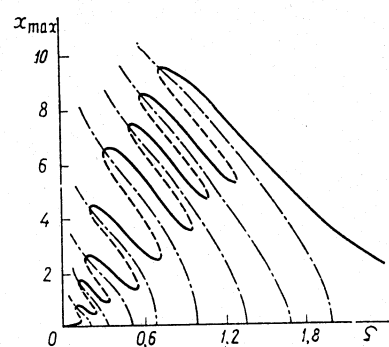


FIG. 2. Typical amplitude-frequency characteristic of Eq. (5). In certain frequency ranges there are several solutions and the initial conditions determine which of the solutions applies (multistability).

obtained by assuming that a solution of Eq. (5) has the form $\sum^i b_i \cos [(m/k)_i \Omega \tau + \varphi_i]$, where m and k are mutually prime numbers, and by applying the harmonic balance condition. However, such a procedure presents considerable difficulties even when finding the skeletal curves. The only feature that has been established relatively easily as that all these curves are inclined in the direction of lower frequencies.

The expected form of the amplitude-frequency characteristic is shown in Fig. 2. The number of branches of the stable solution exceeds two and in the region of overlap of several branches a laser can exhibit multistability. One particular consequence is a complex hysteretic pattern with several loops and not just one as in Fig. 1.

4. STOCHASTIC REGIMES AND THEIR APPEARANCE

In the case of strong modulation ($R \gg 1$) we can expect an irregular stochastic response of a laser to harmonic modulation of the losses.^{13,20} According to the modern theory of oscillations, stochasticity can appear in the course of gradual variation of a parameter of a nonlinear system which produces an infinite chain of bifurcations of the doubling period.^{22,23} This process of gradual loss of the stability by limit cycles of increasing periods is known as a soft scheme of transition to stochasticity. These results have been obtained within the framework of a one-dimensional model, but a soft transition to stochasticity is possible also in more complex systems with a finite number of the degrees of freedom^{11,12,24} and also in distributed systems.⁹ However, even in such a relatively simple system as a parametrically excited oscillator with a cubic nonlinearity it is found that there are independent chains of different elementary periods before chaos is established and a transition from chaos is hard.²⁴ Clearly, the soft regime of stochasticity corresponds to motion along just one branch of solutions, whereas the hard transition to or from chaos is due to switching between branches. This is confirmed in Ref. 20, where a study is reported of the soft transition to stochasticity for both branches of the main resonance and of a hard transition from a stochastic regime corresponding to a lower branch to a periodic regime on an upper branch is predicted.

Numerical investigations of stochastic operation of a laser with periodic loss modulation and of the laws govern-

ing the transitions to such regimes were carried out bearing in mind the above discussion.

The form of the oscillations found by numerical solution of the system (3) can be judged on the basis of the graphs given in Ref. 15, where they are compared also with experimental oscillograms. The difference between the models (a solid-state laser of the ring resonator is considered in Ref. 15 and an allowance is made for all the specific features of the interaction between the opposite waves) is of no importance in the present context.

The nature of motion (periodic or stochastic) is established by the Poincaré mapping method²⁵ in which the successive values of the variable x are found at intervals equal to the modulation period $T = 2\pi/\Omega$. Repetition after n periods indicates the existence of n -periodic motion (i.e., of oscillations with the period nT), which in the phase space corresponds to a stable n -stage limit cycle. The absence of repetition indicates that the solution is nonperiodic. In the latter case we studied also the stability of motion and with this in mind we calculated the characteristic Lyapunov quantity

$$\lambda = \frac{1}{L} \sum_{l=1}^L \ln \left[\sqrt{(x_{1l} - x_{2l})^2 + (z_{1l} - z_{2l})^2} / \Delta \right].$$

Here, $\Delta = [(x_{10} - x_{20})^2 + (z_{10} - z_{20})^2]^{1/2}$ is the initial separation between a path being investigated and a perturbed path, which is selected to be sufficiently small so that the result is independent of the magnitude of the perturbation. This condition that the paths being compared should be adjacent ensures that the gradual increase in the separation between them is exponential and it should be obeyed throughout the period covered by calculation. The positive nature of λ indicates that the paths spread apart and become mixed in the phase space, i.e., that a strange attractor exists. If λ is negative, then the paths are stable against small perturbations and they correspond to periodic oscillations. The case $\lambda = 0$ corresponds to nominally periodic motion of the system.⁴

The prediction of a complex structure (of the type shown in Fig. 2) and the amplitude-frequency characteristic with a large number of branches are supported by the results of a numerical solution of the system (3) obtained for parameters typical of a solid-state laser: $\epsilon_1 = 1.5 \times 10^{-2}$; $\epsilon_2 = 0.75 \times 10^{-2}$; $T_1/T_c = 10^4$; $N_0/N_{th} = 1.5$. The results obtained for $R = 1.8$ are shown in Fig. 3. We can see five

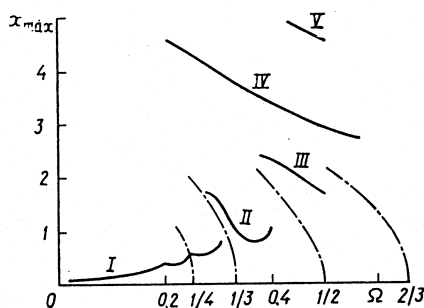


FIG. 3. Dependences of the peak values of the solutions x_{max} on the modulation frequency plotted for $R = 1.8$. The resonances at harmonics and subharmonic of the frequency of the external stimulus are given on the abscissa.

branches of stable solutions identified primarily in accordance with the quantitative criterion which is the value of x_{\max} . The interval $0.27 < \Omega < 0.32$ contains initially three branches and, depending on the initial conditions, we can achieve one of the three stable regimes that correspond to given loss modulation parameters.

In the lowest branch I there are resonances at frequencies $1/5$ and $1/4$. Since there is no reversal of the resonance curves, the lowest curve extends without discontinuities to the $1/3$ resonance, the upper branch of which is II, which is also the lower branch of the resonance $1/2$. It is difficult to classify the branches IV and V; we can simply assume that the branch V is associated with the main resonance. In the range $\Omega > 3.5$ there is only one branch.

Variation of Ω , even that not accompanied by a change in the branch, nevertheless alters the nature of the solutions. The changes which occur may be of two types: greater complexity of the oscillation profile with a constant period (which is not associated with bifurcation); bifurcation doubling of the period associated with a transition to chaos. In this connection it should be pointed out that Fig. 3 shows only segments of the branches corresponding to the one-period solutions.

The complexity of the amplitude-frequency characteristic and the nature of the solutions are governed by the value of R . As R increases, so does the total number of the branches and so does the number of those branches which exhibit bifurcation and autostochasticity. In the case of a low modulation depth corresponding to $R = 0.02$ the resonance curve has just one peak and the laser response is linear throughout the investigated frequency band, with the exception of the range $0.95 < R < 1.3$, where oscillations retain the period of the external force but are not sinusoidal.

The value $R = 1.2$ corresponds to an amplitude-frequency characteristic which includes three branches of stable solutions. The solutions which belong to the lower branch have the period $T = 1$. The middle branch, which corresponds to the resonance $1/2$, has solutions with $n = 2$: they begin from bifurcation for $\Omega = 0.58$, and are present right up to termination of the branch at $\Omega = 0.62$. The upper branch shows, on increase in Ω , a sequence of doubling bifurcations which converge to the critical value $\Omega_c = 1.4$. In the interval $1.4 < \Omega < 1.8$ the laser response is stochastic but, beginning from $\Omega \approx 1.8$, the stochasticity disappears via a reverse sequence of doubling bifurcations, i.e., it disappears by a soft transition.

At large values of R the stochastic regime occurs on more than one branch. In the $R = 1.8$ case discussed above which is predicted by numerical solutions for the two highest of the five branches found by us and the stochasticity regions $0.7 < \Omega < 1.2$, and $1.5 < \Omega < 2.6$ do not overlap on the frequency scale. Moreover, in the case of some values of R in the range $1.2 < \Omega < 3.0$ there is motion with periods which are odd numbers of time (3, 5, 9, 11, 21) greater than the loss modulation period. A more thorough investigation reveals doubling chains on the relevant solution branches, which differ in respect of the elementary period.

A further increase in R complicates greatly the whole pattern because of broadening of the stochasticity zones.

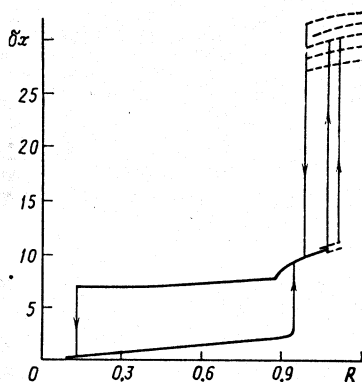


FIG. 4. Dependences of the peak-to-peak amplitude of the oscillations $\delta x = x_{\max} - x_{\min}$ on the depth of modulation plotted for $\Omega = 1.5$. The stochasticity regions are identified by dashed curves.

When $R = 3.8$, the separation into separate branches becomes problematic. However, this is not true of the lowest branch for which the most complex motion is characterized by $n = 8$ and a termination with a transition to the upper branches is localized at $\Omega = 0.3$

The many-sheet structure of the phase of the parameters Ω , R is manifested also in calculations in which the frequency is fixed and the depth of modulation is varied. The dependence $\delta x(R)$ calculated for $\Omega = 1.5$ is shown in Fig. 4. As R is increased from zero, the amplitude of pulsations of the laser radiation rises slowly and at $R \approx 0.95$ this is terminated by a hard transition regime with a larger value of δx . Next, a sequence of bifurcations involving doubling of the period at $R \approx 1.04$ gives rise to a chaotic regime and almost immediately after at $R \approx 1.05$ there is a harder change to another chaotic regime but with deeper minima of x . Figure 4 shows that the dependences $\delta x(R)$ are of multiloop hysteresis nature. Thus, at $\Omega = 1.5$ we can observe simultaneously the following pairs of regimes: two periodic regimes ($0.13 < R < 0.95$); periodic regime—chaos ($1.01 < R < 1.04$); chaos—chaos ($R > 1.04$).

5. CONCLUSIONS

We can summarize the above discussion by drawing the following conclusions. A laser with a delayed-response active medium subjected to a low-frequency (of the order of the relaxation frequency) periodic modulation of the losses is a multistable system with complex hysteretic properties: depending on the initial conditions, at fixed values of the parameters it is possible to observe one of several potential regimes with a characteristic periodicity, depth, and nature of modulation of the laser radiation.

Since the potential applicable to a nonlinear oscillator $V(x) = -x + e^x$ has just one extremum, such multistability cannot be associated with the complex profile of the potential as postulated in Ref. 14. In fact, multistability is due to a complex multiple-branch structure of the resonance response of a nonlinear oscillatory system to an external stimulus.

This confirms the conclusions reached in earlier theoretical and experimental investigations^{13-15,20} of the existence of such sets of parameters which correspond to sto-

chaotic regimes of laser operation, i.e., to singularities of the strange attractor type in the phase space of the system. The dynamic nature of chaos is indicated by the specific sequences of bifurcations of doubling of the laser response period, which precede chaotization of the lasing regime as a result of monotonic variation of the parameters of the external stimulus. The existence of a chain of bifurcations is the most convenient criterion for identification of the dynamic chaos regime under experimental conditions.

A sequence of bifurcations of the period doubling exhibits certain relationships that follow from the theory of Feigenbaum²³ as long as the system is within one branch of a resonance structure. Abrupt transition to another branch results in termination of a Feigenbaum chain of period doubling, which changes to another similar chain but probably with a different elementary period or to a stochastic regime. Therefore, a stochastic regime may be attained as a result of monotonic variation of the parameters either as a result of a soft transition (without a change in the branch) or a hard transition (because of a change to a new branch).

¹⁸The existence of bistability in this laser model was pointed out earlier.¹⁸ However, an incorrect conclusion on the slope of the resonance curves in the direction of higher frequencies was reached in Ref. 18 because of going beyond the limit of validity of the approximation of a nonlinear characteristic by a cubic polynomial.

¹A. Z. Grasyuk and A. N. Oraevskii, Proc. Fourth Intern. Congress on Microwave Tubes, Scheveningen, Holland, 1962, publ. by Centrex, Eindhoven (1963), p. 446.

²A. Z. Grasyuk and A. N. Oraevskii, Radiotekh. Elektron. **9**, 524 (1964).

³E. N. Lorenz, Proc. Intern. Symposium on Numerical Weather Prediction, Tokyo, 1960, p. 629; in: Strange Attractors [Russian translation], Mir, Moscow, (1981), p. 88.

⁴M. I. Rabinovich, Usp. Fiz. Nauk. **125**, 123 (1978) [Sov. Phys. Usp. **21**, 443 (1978)].

⁵A. N. Oraevskii, Kvantovaya Elektron. (Moscow) **8**, 130 (1981) [Sov. J. Quantum Electron. **11**, 71 (1981)].

⁶I. I. Matorin and Ya. I. Khanin, Izv. Akad. Nauk SSSR Ser. Fiz. **45**, 415 (1981).

⁷Ya. I. Khanin and I. I. Matorin, Opt. Quantum Electron. **13**, 439 (1981).

⁸M. Mayr, H. Risken, and H. D. Vollmer, Opt. Commun. **36**, 480 (1981).

⁹C. O. Weiss and H. King, Opt. Commun. **44**, 59 (1982).

¹⁰A. V. Gaponov, M. I. Rabinovich, and M. F. Shapiro, Vestn. Mosk. Univ. Fiz. Astron. **19**, No. 4, 125 (1978).

¹¹T. Yamada and R. Graham, Phys. Rev. Lett. **45**, 1322 (1980).

¹²H. J. Scholz, T. Yamada, H. Brand, and R. Graham, Phys. Lett. A **82**, 321 (1981).

¹³D. V. Ivanov, Ya. I. Khanin, I. I. Matorin, and A. S. Pikovsky, Phys. Lett. A **89**, 229 (1982).

¹⁴F. T. Arecchi, R. Meucci, G. Puccioni, and J. Tredicce, Phys. Rev. Lett. **49**, 1217 (1982).

¹⁵P. A. Khandokhin and Ya. I. Khanin, Kvantovaya Elektron. (Moscow) **11**, 1483 (1984) [Sov. J. Quantum Electron. **14**, 1004 (1984)].

¹⁶Ya. I. Khanin, Laser Dynamics [in Russian]. Sovet-skoie Radio, Moscow (1975).

¹⁷G. N. Vinokurov, N. M. Galaktionova, V. F. Egorova, A. A. Mak, B. M. Sedov, and Ya. I. Khanin, Zh. Eksp. Teor. Fiz. **60**, 489 (1971). [Sov. Phys. JETP **33**, 262 (1971)].

¹⁸V. N. Tsikunov, Zh. Eksp. Teor. Fiz. **58**, 1646 (1970) [Sov. Phys. JETP **31**, 882 (1970)].

¹⁹E. M. Belenov, V. N. Morozov, and A. N. Oraevskii, Tr. Fiz. Inst. Akad. Nauk SSSR **52**, 237 (1970).

²⁰D. V. Ivanov, I. I. Matorin, A. S. Pikovskii, and Ya. I. Khanin, Abstracts of Papers presented at Tenth All-Union Conf. on Coherent and Nonlinear Optics, Erevan, 1982 [in Russian], p. 200.

²¹N. N. Bogolyubov and Yu. A. Mitropol'skii, Asymptotic Methods in the Theory of Nonlinear Oscillations, Gordon and Breach, New York, (1964).

²²A. N. Sharkovskii, Ukr. Mat. Zh. **16**, 61 (1964).

²³M. J. Feigenbaum, J. Statist. Phys. **19**, 25 (1978).

²⁴F. M. Izrailev, M. I. Rabinovich, and A. D. Ugodnikov, Phys. Lett. A **86**, 321 (1981).

²⁵A. A. Andronov, A. A. Vitt, and S. E. Khaikin, Theory of Oscillators, Addison-Wesley, Reading, Mass.; Pergamon Press Oxford (1966).

²⁶V. V. Migulin, v. I. Medvedev, E. R. Mustel', and V. N. Parygin, Fundamentals of the Theory of Oscillations [in Russian], Nauka, Moscow, (1978).

Translated by A. Tybulewicz