From "Nonlinear and Turbulent Processes in Physics: proceedings of the Second International Workshop on Nonlinear and Turbulent Processes in Physics, Kyiv, USSR, 10-25 October 1983",
ed. R. Z. Sagdeev, Gordon and Breach, Harwood Acad. Publ., 1984, vol. 3

SYNCHRONIZATION AND STOCHASTIZATION OF NONLINEAR OSCILLATIONS BY EXTERNAL NOISE

A.S. PIKOVSKII
Institute of Applied Physics, USSR Academy of Sciences, Gorky, USSR

Chaotic oscillations in dynamic systems have recently been intensively discussed. As a criterion for chaos, one usually uses a so-called "sensitivity to initial cinditions," i.e., exponential instability of the trajectories of the dynamic system. In the present paper, this criterion is applied to the case where a dynamic system is under the influence of a random force. We can also obtain cases of stable and unstable dynamics. These regimes may be distinguished in the following way. Consider an ensemble of identical systems, all subject to the same external noise field. Then, if the trajectories converge exponentially, all the systems will eventually be trapped in the same state. This case may be considered synchronization or "phase locking" by an external noise. The other regime is established if the trajectories diverge; in that case, no correlation between the systems is established -- this case may be called stochastization.

As a concrete example, let us consider a two-dimensional system with a limit cycle. In the vicinity of the cycle, equations of motion have, in the action (I) - angle (θ) variables, have the following form:

$$\dot{I} = -\gamma(I - I_0) + q(I,\theta)\cdot f(t), \quad \dot{\theta} = \omega(I) \tag{1}$$

where γ is the damping rate and I_0 and $\omega_0 = \omega(I_0)$ are the amplitude and frequency of the cycle. The external noise $f(t)$ is assumed to be a random sequence of pulses

$$f(t) = \sigma \sum_i \xi_i \delta(t - t_i) \tag{2}$$

where the pulse amplitudes ξ_i and intervals $T_i = t_{i+1} - t_i$ are independent random variables, $\langle \xi \rangle = 1 - \langle \xi^2 \rangle = 0$, and σ measures the noise intensity. We restrict ourselves, following reference 1, to the case

$$q(I,\theta) = I_0 \cos\theta, \quad \omega(I) = \omega_0 \left[1 + \alpha \frac{I - I_0}{I}\right] \tag{3}$$

We can then readily obtain the following mapping for successive values of the action and the angle between pulses:

$$I_{n+1} = I_0 + e^{-\gamma T_n}(I_n - I_0 + \sigma I_0 \xi_n \cos\theta_n) \tag{4a}$$

$$\theta_{n+1} = \theta_n + \omega_0 T_n + \alpha \omega_0 I_0^{-1} \gamma^{-1} (1 - e^{-\gamma T_n}) \times (I_n - I_0 + \sigma I_0 \xi_n \cos\theta_n) \quad (\mathrm{mod}\ 2\pi) \tag{4b}$$

Assuming $\gamma T_n \gg 1$ for all n, we conclude from (4a) that $I_n \simeq I_0$; thus, (4b) can be considered independently

$$\theta_{n+1} = \theta_n + \omega_0 T_n + K\xi_n \cos\theta_n \quad (\mathrm{mod}\ 2\pi) \tag{5}$$

where $K = \sigma d\omega_0 \gamma^{-1}$. The lyapunov characteristic exponent (LCE) λ corresponding to the phase may then be expressed in the following form:

$$\lambda = \overline{\langle \ln |d\theta_{n+1}/d\theta_n| \rangle} = \overline{\langle \ln |1 - K\xi \sin\theta| \rangle} \tag{6}$$

where the bar denotes averaging over the invariant density $w(\theta)$ and the brackets, averaging over the density of ξ. Unfortunately, $w(\theta)$ cannot be written explicitly, so we present here only the estimates of LCE in the limiting cases of small and large noise. For $K \ll 1$,

$$\begin{aligned}\lambda &= \overline{\langle -K\xi \sin\theta - K^2 \xi^2 \sin^2\theta/2 \rangle} + O(K^3) \\ &= K^2 \langle \xi^2 \rangle \overline{\sin^2\theta}/2 + O(K^3) < 0\end{aligned} \tag{7}$$

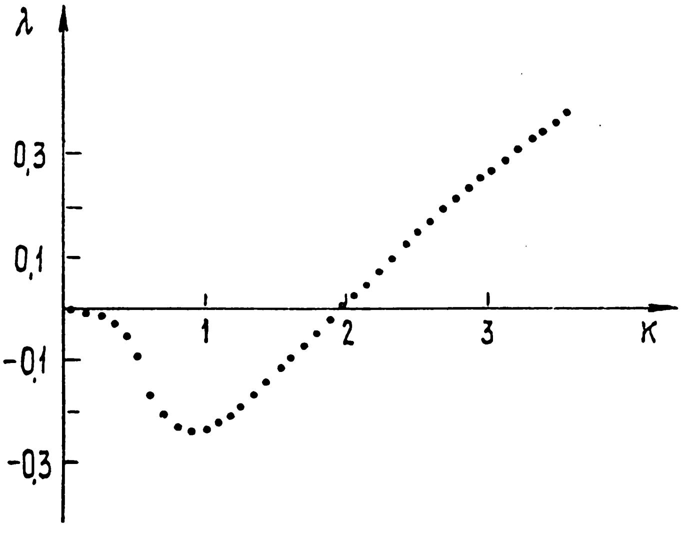

FIGURE 1.

For $K >> 1$

$$\lambda = \langle \ln |K \xi \sin\theta| \rangle = \ln |K| + \langle \ln |\xi| \rangle + \overline{\ln |\sin\theta|} > 0 \qquad (8)$$

Thus, there exists a critical noise intensity beyond which the system is sensitive to initial conditions. The results of numerical modeling of the mapping (5) presented in figure 1 are in agreement with (7,8). The transition to chaos occurs at $K = K_c \simeq 2$. This means that, for $K < K_c$, an ensemble of oscillators (1) will be synchronized by the external noise, and, for $K > K_c$, the phases of all oscillators in the ensemble will be independent. This is confirmed by our numerical calculations.

REFERENCE

1. G.M.Zaslavsky, Phys. Lett., 69A, 145 (1978).