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S. P. Novikov

L. D. Landau Institute of Theoretical Physics
Academy of Sciences of the U.S.S.R., Moscow

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Stochastic Behavior of Dissipative Systems

A. S. PIKOVSKII, AND M. I. RABINOVICH
Applied Physics Institute, Academy of Sciences of the U.S.S.R. 603600 Gorky, U.S.S.R.

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I. Introduction

The mathematical description of a whole variety of phenomena in various branches of fundamental and applied science—physics, chemistry, biology, electronics, ecology, economics, etc., is done using systems of ordinary differential equations. Given definite initial conditions, one can (at least, in principle) obtain the solution for all times (if the system is good enough) and thus adequately describe the phenomenon under study. This approach—the description of a phenomenon by means of an appropriate dynamical system, has demonstrated its fruitfulness for the most varied systems, starting from the dynamics of variable stars (Cepheids) and ending with the dynamics of biological populations. There has even appeared a special branch of science, the theory of vibrations, at the junction between natural science and mathematics. A beautiful general introduction to the theory of vibrations is contained in the book of Andronov, Witt and

Khaikin (1937). In it, we may say, is presented the first stage of the development of this science, when the theory of regular vibrations was constructed.

But in many systems, along with regular behavior, one observes irregular, nonrepeating, chaotic behavior. For a long time these phenomena were ignored by the theory, since it was assumed that in good dynamical systems, because of their complete predictability, one can observe only regular motion. This state of affairs has changed fundamentally over the last ten years. Thanks to the work of Lorenz (1963), Ruelle and Takens (1971), Martin and McLaughlin (1975), as the achievement of a wide circle of investigators, there developed the concept of strange attractors, of stochastic vibrations in dynamical systems. It appeared that irregular practically unpredictable behavior can be observed in simple systems in three-dimensional phase space. At present the number of papers in this field is growing continuously, and the field of investigation is both expanding (as more and more models appear from different fields) and also becoming more profound (as we achieve ever greater understanding of the basic models).

In this survey we shall try to present the main accomplishments of Soviet investigators in this area. We shall restrict our survey to dissipative systems; a survey of work on hamiltonian systems can be found in B. V. Chirikov (1979). Just as for nonlinear hamiltonian mechanics, which studies the regular behavior of dynamical systems, there remained for a long time the challenge of the problem of the foundation of statistical mechanics, so for the theory of self-oscillations the challenge was the problem of the development of turbulence. The fact that turbulence is a stochastic process is clear from its definition. Less obvious is that turbulence is a self-oscillation. The picture of turbulence as self-oscillations in a continuous medium was the basis of the Landau theory of the origin of turbulence. At the time (1944) when Landau developed this theory, in the theory of vibrations only one type of self-oscillation was known, periodic oscillations to which there corresponds a limit cycle. On this basis, Landau assumed that the transition to turbulence with increasing Reynolds number Re consists in successive complication of the self-oscillations; at Re_1 a limit cycle with frequency ω_1 appears, at $Re_2 > Re_1$ a mode with frequency ω_2 becomes unstable and a doubly periodic auto-oscillation appears, then more new frequencies, etc., and the whole complex self-oscillation that develops as a result of the chain of bifurcations is turbulence. Actually, Landau proposed for describing turbulence an attractor in the form of a torus with a quasiperiodic winding. Of course such an attractor is more complicated than a limit cycle, but we know now that there are even more complicated attractors—strange attractors. A notable fact is that the bifurcational development of many strange attractors is similar to Landau's description. In fact, Landau assumed that the frequencies of the "modes" $\omega_1, \omega_2, \ldots,$ are not related. But in a nonlinear system there is a connection: in the presence of the frequency ω_1 the most unstable mode is the one with frequency $\omega_1/2$ (the fundamental band of parametric instability). As a result of such a resonance bifurcation, as was proved rigorously by Yu. I. Neimark (1959), the initial limit cycle becomes unstable, and in its neighborhood there appears a stable cycle with double the period. There can be infinitely many such bifurcations with doubling of the period, and they actually lead to the appearance of the strange attractor. Thus Landau's hypothesis of the transition to turbulence as the result of successive complication of self-oscillations finds a confirmation also within the framework of strange attractors.

Just as the theory of stochastic vibrations in hamiltonian systems was not restricted to the problem of the foundation of statistical mechanics, but found wide application in plasma physics, celestial dynamics, etc., so the theory of stochastic self-oscillations went far beyond the problem of turbulence and is used by many branches of science. The main subject of this theory is the study of simple models —nonlinear equations, individual parts of which are given physical (or chemical, biological, etc.) meaning. The technique of investigation —a combination of rigorous mathematical, qualitative computational, experimental, and other methods, is based on the traditions of the classical theory of vibrations, which is associated with the names of A. M. Lyapunov, L. I. Mandel'shtam, A. A. Andronov, I. M. Krylov, N. N. Bogolyubov, et al.

It is this approach—the investigation of simple models and development on this basis of intuition and "physical" pictures that we use in the present work.

In Sec. II we present the results of studies of the simplest dissipative systems with stochastic behavior, and discuss the fundamental mechanisms for the appearance of chaos. In Sec. III we interpret some experiments using these mechanisms.

II. SIMPLE SYSTEMS

1. Systems Close to Conservative

The main problems which arise in physical investigations of stochasticity of dynamical systems are the development of physical pictures of the "structure" of stochasticity, the determination of criteria for its appearance and the working out of approximate methods of description. Here the use of the accumulated experience with hamiltonian systems seems attractive (cf., for example, the paper of Chirikov, 1979). It should be emphasized that if we disregard transient processes and consider only stationary behavior, then the attractor in the dissipative system and the invariant manifold in the hamiltonian system will be analogous. This also applies to simple regimes, like the periodic one (a limit cycle is analogous to a closed trajectory of a hamiltonian system), and to the hyperbolic regime (or strange regime, as it is now called). We note however, that in real systems one observes, not "pure" stochasticity, corresponding to an invariant manifold, or to a hyperbolic (or almost hyperbolic) attractor, but rather a regime in which there participate to some extent transient trajectories or neighboring invariant manifolds (for a more detailed discussion, cf. later).*

One of the simplest hamiltonian systems with stochastic behavior is the nonlinear oscillator on which a periodic external force acts (cf., for example, Zaslavskii and Chirikov, 1971). Zaslavskii (1978) has studied the effect on this system of nonconservative additional terms. His model, in action-angle variables I, θ, is described by the following equations:

$$\begin{aligned} \dot{I} &= -\gamma(I - I_0) + \epsilon I_0 \cos\theta f(t) \\ \dot{\theta} &= \omega_0 + \alpha\omega_0(I - I_0)/I_0 \end{aligned} \tag{2.1}$$

Here γ^{-1} is the time (for zero perturbation ϵ) of relaxation to a limit cycle, having amplitude I_0 and period $2\pi/\omega_0$; α is the parameter of nonlinearity of the oscillator, ϵ the amplitude of the external force. The simplest results are obtained if we take the external force $f(t)$ in the form of a periodic succession of narrow pulses

$$f(t) = \sum_{h=-\infty}^{\infty} \delta(t - hT) \tag{2.2}$$

*This regime may be called "wild," following Newhouse (1978).

Then Eq. (2.1) can be integrated over the interval T from pulse to pulse, and thus we reduce the problem to the mapping of the cylinder ($0 < I < \infty$, $0 \leqslant \theta < 2\pi$) onto itself:

$$
\begin{aligned}
I_{n+1} &= I_0 + (I_n - I_0)e^{-\gamma T} + I_0 \epsilon \cos\theta_n \\
\theta_{n+1} &= \theta_n + \omega_0 T\left(1 + \mu \frac{I - I_0}{I_0}\right) + \epsilon\mu\alpha\omega_0 T \cos\theta_n \qquad (\mathrm{mod}\, 2\pi)
\end{aligned}
\tag{2.3}
$$

where $\mu = (1 - \exp(-\gamma T))/\gamma T$.

If there are no nonconservative elements ($\gamma = 0$, $\mu = 1$), the system (2.3) goes over into the well-investigated "fundamental model of stochasticity of hamiltonian systems" (Zaslavskii and Chirikov, 1971). Here the parameter $K = \epsilon\alpha\omega_0 T$ plays the fundamental role. When $K \gtrsim 1$ the motion is stochastic, which corresponds to a spreading of the phase of the oscillator: $d\theta_{n+1}/d\theta_n \sim K$. In the dissipative case the condition for spreading out of the phase reduces to $d\theta_{n+1}/d\theta_n \sim K\mu \gtrsim 1$. Actually one can here introduce the parameter $K' = \epsilon\alpha\omega_0 \min(\gamma^{-1}, T)$, reflecting the fact that the spreading of phase ceases if the system relaxes to a limit cycle. Thus, for strong damping ($\gamma/\omega_0 \approx 1$), for the appearance of stochasticity one must have a strong nonlinearity α of the oscillator and strong perturbation ϵ.

The external form of the strange attractor that appears in the system (2.3) when $\epsilon = 0.3$, $\alpha = 0.3$, $\epsilon\alpha\omega_0 T = 9.03$, $\gamma T = 5$ is shown in Fig. 2.1. Here we see clearly the characteristic foliation along the unstable direction (θ). It is characteristic for many systems (the most popular is the Hénon mapping, 1976), and gives evidence of a Cantor transverse structure.

Of definite interest is the study of the transition from hamiltonian

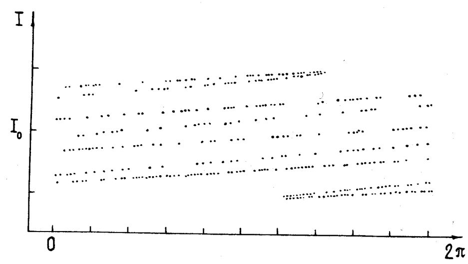


Figure 2.1. Foliation of the strange attractor of system (2.1).

stochasticity ($\gamma = 0$) to a developed strange attractor (larger γ). Such a numerical experiment was done by Izrailev and Chirikov (1974) on the following simple model

$$P_{n+1} = \{P_n + Kf(x_n) - \gamma(P_n - 1/2)\}$$
$$x_{n+1} = \{x_n + P_{n+1} - 1/2\} \tag{2.4}$$

where { } denotes the fractional part of the number. The function $f(x)$ was taken equal to $x^2 - x + 1/6$.

In the conservative approximation ($\gamma = 0$) the behavior of the system (2.3) is determined by the parameter K, and when $K \gtrsim 1$ will be stochastic. A characteristic feature of systems of this type (including system (2.3) when $\gamma = 0$) is that the stochastic component does not fill all of the phase space (in this case the torus $0 \leqslant p < 1$, $0 \leqslant x < 1$). Ergodic regions have neighboring regions of regular stable quasiperiodic behavior, due to critical points of the function $f(x)$, where $df/dx = 0$. Since in the dissipative case it is not just the existence of a stochastic manifold, but also its attraction that is important, the effect of introducing dissipation is not trivial. The numerical experiment of Izrailev and Chirikov showed that, for small γ, in place of the islands of stability of the hamiltonian system there appear stable limit cycles, to which sooner or later almost all the initial points are attracted. It appeared that the number of steps N until capture (the "lifetime" of stochasticity) is described approximately by the formula

$$N \approx (S\gamma)^{-1} \tag{2.5}$$

where S is the area of the stable regions. Formula (2.5) is valid only for sufficiently small damping γ; when $\gamma \approx 0.1$, N increases sharply and approaches infinity—a point wanders for an infinitely long time (within the limits of the computer) over the stochastic region, which now takes on the foliation characteristic of strange attractors (cf. Fig. 2.1).

It will be appropriate here to discuss the relation between "complete" and "incomplete" stochasticity in hamiltonian and dissipative systems. By "completeness" of the stochastic situation we mean the fulfillment of the condition of hyperbolicity for all the phase space of the hamiltonian system (or for the attractor). This condition was first introduced by Anosov and later was transformed by Smale into axiom A. Roughly speaking, it means a constant instability of the trajectories of the dynamical system: at any point of a trajectory there is a direction along which nearby trajectories run away from it, and under the action of the time these directions go over into one another. In the conservative case the hyperbolicity condition is satisfied by, for example, the motion of a particle on a billiard table with walls that are everywhere concave inward, or a system of hard spheres in a rectangular box. Among real dissipative systems we may cite only that of Lorenz (1963), which, however, is only "weakly" hyperbolic. There are also abstract mathematical constructions of attractors satisfying axiom A (Smale, 1967; Plykin, 1974), but they have not yet been discovered in real systems. Completely stochastic systems are subject to mathematical investigation; for them, in particular, one has established the existence of an invariant probability measure, exponential falloff of the correlation function, etc.

The structure is much more complicated for a system with "incomplete" stochasticity, for which the instability of neighboring trajectories is not constant: unstable directions can change to stable. In the hamiltonian case (for example, the systems (2.3), (2.4) with $\gamma = 0$), this is reflected in the fact that the stochastic region does not occupy the whole of the phase space, but is bordered by regions of regular behavior. In dissipative systems such breakdowns of hyperbolicity lead to the result that stable limit cycles appear, their number, as follows from the results of Gavrilov (1973) and Newhouse (1978), can even be infinite. These cycles are not observed because in a numerical experiment (and, even more, in reality) there are noises everywhere which push the trajectory into the neighboring (apparently nonattracting) stochastic region. The statistics of the noise have practically no effect on the observed picture, since during the time of wandering over the stochastic region, because of instability of the trajectories, the characteristics of the noise are rapidly forgotten. Apparently, in most real systems (including (2.3) and (2.4)) there is realized such an incompletely stochastic strange attractor.

It is difficult to treat mathematically such systems with broken hyperbolicity.

The greatest interest has been in studying the simplest models: a continuous mapping of the segment $x_{i+1} = f(x_i, \lambda)$, having for example, the form $f = \lambda x(1 - x)$. Yakobson (1980) proved that there exists a set of values λ with measure greater than zero for which this mapping has a continuous invariant probability distribution. This permits one to expect that for neighboring values of λ the stochastic description of this mapping is adequate.

The investigation of stochasticity by reduction to a one-dimensional mapping has now been used extensively, and has, in particular, made possible considerable progress in the study of the Lorenz attractor. The point is that the property of instability of trajectories plays a decisive role in stochasticity, and if the unstable direction is the same everywhere, the whole problem reduces to a one-dimensional mapping. Still it is not excluded that the unstable direction be multidimensional, when the picture is complicated significantly (cf. Rössler, 1979).

2. *Homoclinic Structure*

By a homoclinic structure we mean a set appearing in the neighborhood of a homoclinic trajectory—a trajectory doubly asymptotic to an unstable equilibrium state or an unstable cycle. In three-dimensional phase space an unstable cycle has two-dimensional invariant surfaces, a stable one and an unstable one. If the unstable manifold of the one cycle intersects the stable manifold of the other, the curve of their intersection is a trajectory that runs from the first cycle to the second, i.e., precisely a homoclinic trajectory. The intersection of two-dimensional surfaces in three-dimensional space is not destroyed by small perturbations, so the homoclinic situation is structurally stable.

The fact that the motion in the neighborhood of a homoclinic trajectory is exceptionally complex was already noted by Poincaré in connection with the three-body problem. Later Smale showed that in this neighborhood there is a countable set of unstable cycles. The complete description of the homoclinic structure, based on the methods of symbolic dynamics, was given by Shil'nikov (1967).

We should mention the essential role that homoclinic structures play in the building of a stochastic attractor. In fact, there are unstable cycles in any stochastic attractor. The stable and unstable manifolds of these cycles intersect, forming a homoclinic structure. This homoclinic structure, in turn, contains unstable cycles, the stable and unstable manifolds of which form homoclinic structures of the second order, and so on. Thus the fine structure of the strange attractor is "larded" with homoclinic structures, forming an infinite hierarchy. However, the object of most interest is the top of this hierarchy, that generates all the rest. This is the object that we will consider.

It is easiest to understand the homoclinic structure on the example

of a nonautonomous system of second order of the form

$$\dot{x} = f(x, y) + \epsilon f_1(x, y, \omega t)$$
$$\dot{y} = g(x, y) + \epsilon g_1(x, y, \omega t) \quad (2.6)$$

where f_1, g_1, are periodic functions of ωt. For concreteness we consider an example proposed by Batalova and Neimark (1972). They assumed that the system (2.6) is close to hamiltonian:

$$\dot{x} = \frac{\partial H}{\partial y} - \alpha H \frac{\partial H}{\partial x} + \beta y \sin \omega t$$
$$\dot{y} = -\frac{\partial H}{\partial x} - \alpha H \frac{\partial H}{\partial x} + \beta y \sin \omega t \quad (2.7)$$

where

$$H(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2)$$

When $\alpha = \beta = 0$ the system is autonomous and conservative, its hamiltonian is H; the trajectories of this system are shown in Fig. 2.2. The perturbation consists of two parts. The term with β is a periodic perturbation which leads to splitting off of separatrices (curves with $H = 0$) (Fig. 2.3) and the formation of an ergodic layer. The term with α is essentially dissipative. Its role can be understood by setting $\beta = 0$, when $dH/dt = -\alpha H((H_x)^2 + (H_y)^2)$, i.e., the separatrices become an attractor (Fig. 2.4). If $\alpha \neq 0$ and $\beta \neq 0$ we may hope that because of the smallness of the perturbation these terms act independently, and as a result the ergodic bundle becomes attractive (or, conversely, the attracting separatrices are stochastized). Batalova and Neimark (1972) studied the resulting stochasticity numerically for $a = 0.5$, $\alpha = \beta = 0.1$,

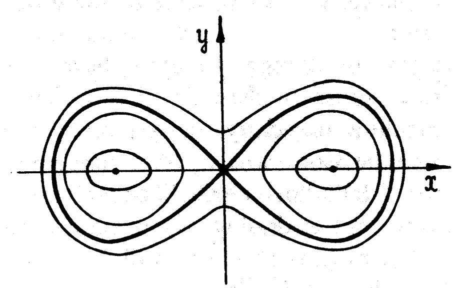


Figure 2.2. Phase portrait of system (2.7) for $\epsilon = 0$ and $f = \partial H/\partial y$, $g = -\partial H/\partial x$, where $H(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2)$.

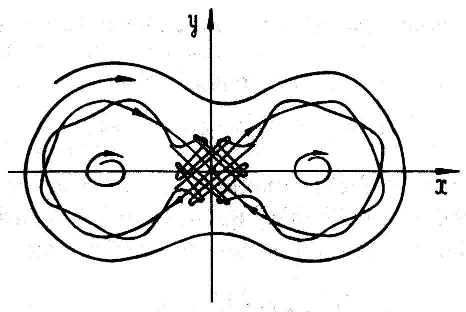


Figure 2.3. Splitting off of separatrices of the nonlinear oscillator (2.7) for $\epsilon \neq 0$.

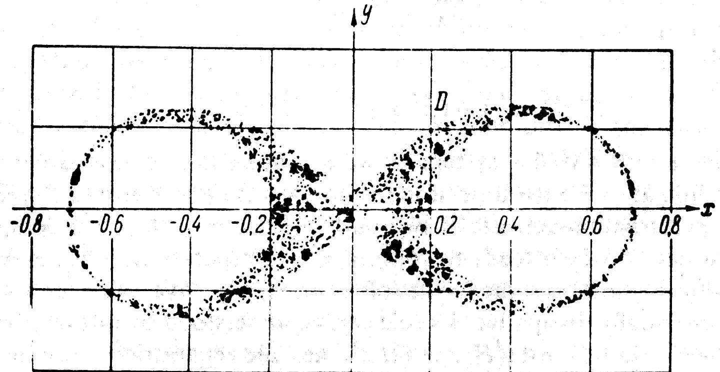


Figure 2.4. Poincaré mapping for the oscillator shown in Fig. 2.2 in the presence of a perturbation (cf. (2.7): $\epsilon = 0.1$, $\omega = 1$, $a = 0.5$.

$\omega = 1$. The Poincaré map $(x, y, t = 0) \to (x, y, t = 2\pi)$ obtained by them is given in Fig. 2.4. But the attractor based on the homoclinic structure is not completely stochastic, since periodic stable trajectories can exist in its interior.

An attractive feature of strange attractors, based on a homoclinic structure, is that to establish stochasticity it is sufficient to show the existence of a homoclinic trajectory. In systems of the type of (2.6) when $\epsilon \ll 1$ this can be done analytically, using the perturbation method developed by Mel'nikov (1963). Suppose that when $\epsilon = 0$ Eq. (2.6) has a trajectory that is doubly asymptotic to the equilibrium state. When $\epsilon \ll 1$ separatrices are "split off," the separation Δ between stable and unstable manifolds of the periodic motion (which in a nonautonomous system replaces the state of equilibrium) is calculated by perturbation methods: $\Delta = \epsilon\Delta_1(t_0) + \epsilon^2\Delta_2(t_0) + \cdots$, where t_0 is the phase of the functions f_1, g_1. If $\Delta_1(t_0)$ is a periodic function of t_0, vanishing at the point t_0^* and $d\Delta_1(t_0^*)/dt_0 \neq 0$, then there follows the existence of a rough homoclinic trajectory, and consequently a stochastic homoclinic structure. Mel'nikov's method has been applied to a variety of systems. Morozov (1973) investigated the nonlinear oscillator with damping and an external force

$$\ddot{x} + x - \alpha x^3 = \epsilon[-\delta\dot{x} + \sin\omega t] \tag{2.8}$$

Here the expression for $\Delta(t_0)$ takes the form

$$\Delta(t_0) = -\frac{\omega\pi\sqrt{2}\cdot\sin\omega t_0}{\sqrt{\alpha}\,\sinh\dfrac{\pi\omega}{\sqrt{2}}} + \frac{2\sqrt{2}}{3\alpha}\delta \tag{2.9}$$

It appears that the damping results in a nonoscillatory contribution to the separation between the manifolds, and prevents the appearance of stochasticity, which can occur only for small damping $\delta < (3\pi\sqrt{\alpha}\,\omega/\sinh(\pi\omega/\sqrt{2}))$. A similar system with a different form of the nonlinearity

$$\ddot{x} - x + \alpha x^3 = \epsilon[-\delta\dot{x} + \sin\omega t] \tag{2.10}$$

was studied in detail by Holmes (1979). Thus stochasticity based on a homoclinic structure appears if a periodic external force acts on a nonlinear oscillator, which in the autonomous nondissipative case has coincident separatrices of the equilibrium states. This stochasticity, which is apparently incomplete, can be made attracting if we introduce dissipation. In concluding this paragraph we present a physically interesting example of such a situation. Belykh, Pedersen and Soerensen (1976) investigated the regime in a point Josephson contact, on which a monochromatic external field acts. In dimensionless variables the equation for the phase φ has the form

$$\ddot{\varphi} + \sin\varphi = \rho + \alpha\sin\Omega t - \beta(1 + \epsilon\cos\varphi)\dot{\varphi} \tag{2.11}$$

Here ρ is the external constant current, $\alpha\sin\Omega t$ is the variable external current, and the term with β corresponds to the dissipation. A detailed analysis of Eq. (2.11) showed that in it, along with periodic regimes (synchronization of the Josephson contact) there are possible stochastic regimes, and their appearance is connected with the occurrence of homoclinic structure.

Thus, using the small parameter ϵ, in systems of the type of (2.6), one can establish analytically the existence of a stochastic manifold. There is still another class of systems in which one can, by perturbation methods, rigorously prove the presence of chaos. This is a system with a small parameter on the derivative; we consider it in the next paragraph.

3. *Relaxational Chaos*

Rössler was the first to point out (1976) the possibility of more or less rigorous description of stochastic relaxation oscillations by reduction to a one-dimensional mapping. We shall first describe a very simple situation where one does not even have to solve the differential equations.

Let us consider a relaxing system whose state is completely defined by one variable $x(t)$. In the interval $(0, \theta)$ x varies according to the linear law $\dot{x} = -\lambda$ (slow motion). If at the time t^* the value $x(t^*) = 0$ is reached, then the system goes over abruptly to the state $x(t^* + 0) = \theta$ (fast motion or jump), after which the slow motion begins again, etc. For $\theta = \theta_0 = \text{const}$, the system undergoes a periodic broken oscillation with period $T = \theta_0/\lambda$ (Fig. 2.5). This very simple vibrational system is related to electronics and biology (L. Glass and M. C. Mackey, 1978).

Now suppose that there acts on the system a periodic external force with period τ, resulting in a modulation of the upper threshold $\theta = \theta_0 + \theta_1 \cdot f(t/\tau)$. Then the jump will always occur at different

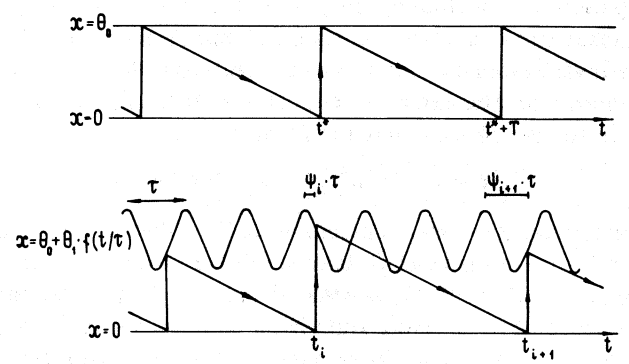


Figure 2.5. Simplest relaxation oscillator under the influence of a periodic force.
a. autonomic relaxation oscillator.
b. periodic change of the upper threshold.

phases of the external force. It is easy to check that if t_i, t_{i+1} are the times of successive jumps, then their phases $\psi_i\{t_i/\tau\}$ where $\{\ \}$ denotes the fractional part (it is assumed that $0 \leqslant \psi < 1$) are related by the transformation (Fig. 2.6):

$$\psi_{i+1} = \{\psi_i + T/\tau + Kf(\psi_i)\} \tag{2.12}$$

Here $K = \theta_1/\tau\lambda$ is the basic parameter determining the regime of the mapping (2.12) and consequently the regime of the relaxer. When $K \gg 1$ the mapping is highly expanding and the system functions in the stochastic regime. Whether this will be "total" or "incomplete" stochasticity depends on the existence of critical points $df/d\psi = 0$ for the function $f(\psi)$. For example, if $f = \sin(2\pi\psi)$, stochasticity will be incomplete; but if $f = \psi$ (i.e., the signal matches the relaxer) then we obtain an everywhere expanding, completely stochastic mapping.

Another example of stochastization of relaxation oscillations under the action of an external periodic force was considered by Rozhdestvenskii (1979). He investigated the forced oscillations of an oscillator with solid friction:

$$\ddot{x} + \omega_0^2 x + F(\dot{x}) = f(t) \tag{2.13}$$

where the nonlinearity $F(\dot{x})$ has the form $F(\dot{x}) = F_0 \operatorname{sgn}(\dot{x}) - h\dot{x}$. The lack of smoothness of $F(\dot{x})$ is related to the presence of a band of insensitivity, and leads to a nonsmooth mapping of the segment onto itself, to which (2.13) reduces for an external force $f(t)$ having the form of a meander.

Let us now consider in more detail an example of stochastic

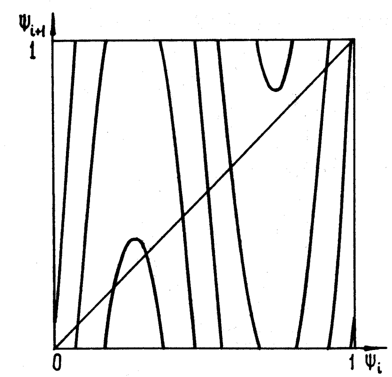


Figure 2.6. Relation of ψ_{i+1} to ψ_i for the simplest relaxation oscillators in the nonautonomic regime; $\{T/\tau\} = 0.1$, $k = 2$, $f(\psi) = \sin(2\pi\psi)$.

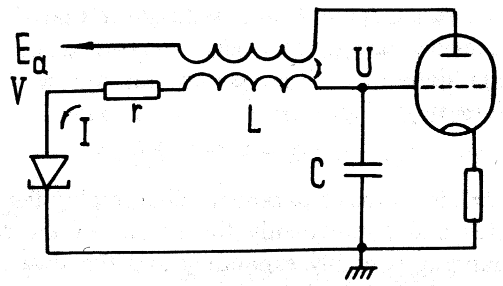


Figure 2.7. Circuit diagram of electronic generator of stochastic oscillations.

relaxation oscillations in an autonomous system, the electronic generator whose circuit is shown in Fig. 2.7 (Kiyashko, Pikovskii, Rabinovich, 1979). It differs from the usual sine-wave generator with grid circuit coupling only in having a tunnel diode connected in series with the inductance. The operation of the circuit is described by the following equations:

$$\begin{aligned} C\dot{U} &= -I \\ LC\dot{I} &= (MS - rC)I - C(U - V) \\ C_1\dot{V} &= I - I_{\text{t.d.}}(V) \end{aligned} \tag{2.14}$$

Here C_1 is the capacity of the tunnel junction, while S is the tube transconductance. The tube characteristic are assumed to be linear; this is justified by the fact that the oscillations are limited by the nonlinear characteristic of the tunnel diode $I_{\text{t.d.}}(V)$ (Fig. 2.8), before the effects of the tube nonlinearity can be seen. We change to dimensionless variables $x = I/I_m$, $y = \sqrt{C}\,U/\sqrt{L}\,I_m$, $z = V/V_m$,

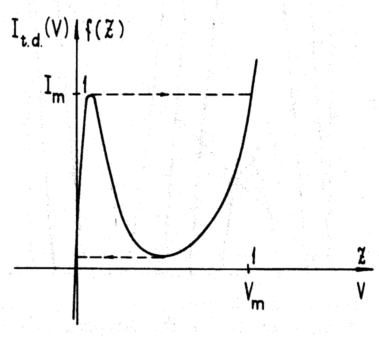


Figure 2.8. Volt-ampere characteristic of tunnel diode.

$\tau = t/\sqrt{LC}$. We then get the system of equations:

$$\begin{aligned} \dot{x} &= 2hx + y - gz \\ \dot{y} &= -x \\ \epsilon\dot{z} &= x - f(z) \end{aligned} \tag{2.15}$$

Here $h = 0.5(MS - rC)(LC)^{-1/2}$ is the increment of growth of oscillations in the circuit, $g = V_m\sqrt{C}/I_m\sqrt{L}$, $\epsilon = gC_1/C$, $f(z) = I_{\text{t.d.}}(V_m z)/I_m$ is the normalized characteristic of the diode.

Usually the capacity of the tunnel diode is small, so that $\epsilon \ll 1$. Consequently the system (2.15) has a small parameter in the derivative and the oscillations will be relaxational. All the trajectories in the three-dimensional phase space consist of fast portions—the lines $x = \text{const}$, $y = \text{const}$, $\dot{z} = O(1/\epsilon)$, and slow sections, lying on the stable branches of the surface $x = f(z)$. An approximate picture of the phase space of the system (2.15) is shown in Fig. 2.9. The single equilibrium state $x = y = z = 0$ is unstable when $h > g/2f'(0)$. Trajectories lying on the slow surface A, speed up around an unstable focus and finally reach the edge of the surface A. Here there is a rapid jump to the slow surface B. Proceeding along B, the trajectory again hops to the surface A, etc.

We now go over from a continuous time description to a discrete one. We construct the Poincaré map of the halfplane $x = 0, y > 0$ on itself. If we set $\epsilon \to 0$, this halfplane is intersected only by trajectories lying on the surface of slow motions, so that we get a mapping T of the half line $y > 0$, $x = z = 0$ into itself: $y \to Ty$. In the general case of

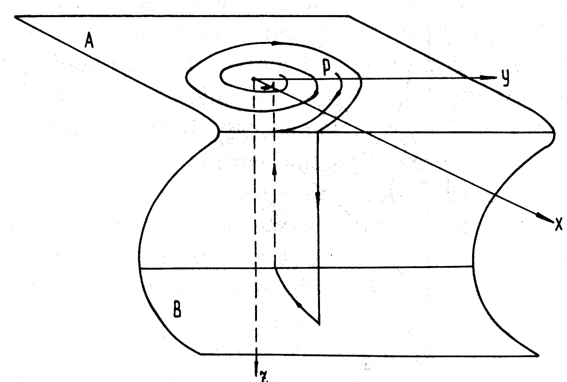


Figure 2.9. Phase space of system (2.15).

a nonlinear function $f(z)$ we cannot describe the mapping T analytically. But a fair approximation can be gotten by approximating $f(z)$ by the piecewise linear function:

$$f(z) = \begin{cases} \alpha^{-1} z & \text{for } z < \alpha \\ \dfrac{1-\alpha-z}{1-2\alpha} & \text{for } \alpha < z < 1-\alpha \\ \dfrac{z-1+\alpha}{\alpha} & \text{for } 1-\alpha < z \end{cases} \tag{2.16}$$

In this approximation A and B are halfplanes, the equation of slow motions on them having the form

$$\begin{cases} \dot{x} = 2\nu x + y \\ \dot{y} = -x \end{cases} \text{if } x, y \in A, \quad \begin{cases} \dot{x} = 2\nu x + y - b \\ \dot{y} = -x \end{cases} \text{if } x, y \in B \tag{2.17}$$

where $\nu = h - \alpha g/2$, $b = g(1-\alpha)$. The Eqs. (2.17) are linear, and by using them we can obtain an explicit form for the mapping T, joining the parts of the trajectory lying on the planes A and B. We shall not describe here the messy formulas, but present only the splitting up of the plane of the parameters (b, ν) into regions of different types of behavior of the mapping T (Fig. 2.10) and discuss the main features of the resulting one-dimensional mappings. These mappings have a point of discontinuity p, the result of the idealization $\epsilon = 0$. If $\epsilon > 0$, and $f(z)$ is a smooth function, the Poincaré map will be continuous. However, near the point p, $dTy/dy \sim \exp(\epsilon^{-1})$, since the trajectory goes along the unstable slow manifold. Thus the jump approximation

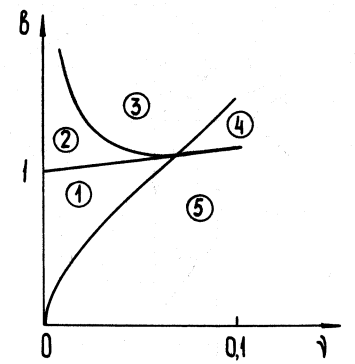


Figure 2.10. Division of plane of system parameters (b, ν) of (2.17) into regions with different dynamics.

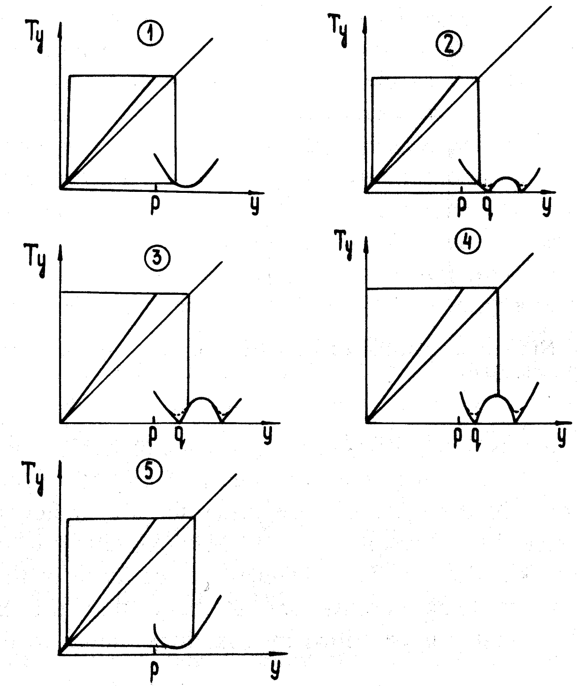


Figure 2.11. Different types of Poincaré mapping of system (2.17) corresponding to regions 1–5 of Fig. 2.10.

is quite good. Another singular point of the mapping, the point q, appears because we chose the function $f(z)$ so that $f(z) = 0$ when $z = 1 - \alpha$, i.e., the quilibrium state lies precisely under the line of the break. If we take into account that $f(z) > 0$ when $z > 0$, the kink at the point q disappears, as shown by the dotted curve in Fig. 2.11. Figure 2.12 shows the Poincaré map T obtained from a numerical study of Eq. (2.15) with the parameters $h = 0.074$, $g = 2.8$, $\epsilon = 0.004$; the function $f(z)$ was approximated by the curve $z \exp(3.61 - 13.5z) + \exp(6.5(z-1))$. It is apparent that the mapping is discontinuous, which is not surprising, since from our estimate its slope is $\sim 10^{100}$. It is shown in detail in Fig. 2.11.

For small ν the critical points of the Poincaré mapping, at which $dTy/dy = 0$, do not belong to the attractor, which is shown dotted in Fig. 2.11. So for these values of the parameters the stochasticity is complete (to order ϵ). For other parameter values there are critical points inside the attractor, so the stochasticity is incomplete. In

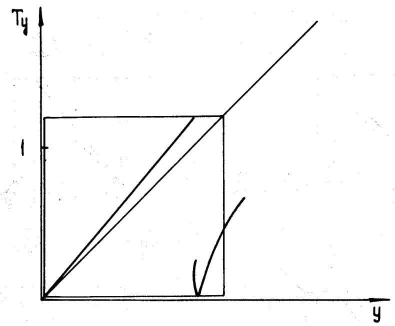


Figure 2.12. Numerically constructed Poincaré mapping of system (2.15) for $h = 0.074$, $g = 2.8$, $\epsilon = 0.004$.

particular the possibility arises of the existence of stable limit cycles—regions of periodic generation. The periodic and stochastic oscillations observed in the electric noise generator are analogous to the regimes observed by Hudson, Hart and Marinko (1979) in the Belousov-Zhabotinskii reaction. The considerable similarity of the regimes of these two systems permits one to hope that the Belousov-Zhabotinskii reaction is described by equations similar to the system (2.15).

In the example described above we have shown that the dynamics of relaxation oscillations reduces, to high accuracy, to a one-dimensional mapping. This was possible because the stable and unstable directions are obtained directly from the equations. In many other cases the structure of the stochasticity is essentially one-dimensional, but this becomes obvious only from numerical studies. Among such situations are, in particular, coupled nonlinear anti-damped and damped oscillators and waves, which we discuss in the following paragraph.

4. *Chaotic Energy Exchange Among Oscillators*

The resonance interaction of waves (oscillator modes) is the most characteristic manifestion of the nonlinear properties of all sorts of media. The nonlinear phenomena arising from such interactions: generation of harmonics and subharmonics, self-modulation and self-focusing of waves, parametric processes of various kinds, are detected in dispersive media even for very small nonlinearity, if the condition

of synchronism is satisfied: $\sum n_i\omega_i = 0$, $\sum n_i\mathbf{K}(\omega_i) = 0$, where ω_i are the frequencies, and $\mathbf{K}(\omega_i)$ are the wave vectors of the interacting waves. The amplitudes of these waves are slowly varying functions of the space coordinates and the time. Nonlinear interaction of quasiharmonic waves plays an important part in plasma physics, hydrodynamics, nonlinear optics, condensed matter physics, and other branches of "nonlinear physics." If the number of elementary excitations in the medium is very large, then, as a rule (an exception, for example, is the case of nonlinear stationary waves), an irregular (turbulent) behavior of the wave field develops. In the absence of energy sources or sinks the spectrum of such "turbulence" corresponds to a uniform distribution of energy over the degrees of freedom (Rayleigh-Jeans distribution). For a self-consistent description of real wave turbulence one must include dissipation and pumping of energy from the source—the external field in heating of a plasma, the wind for the case of waves on water, etc. With such a description, the problem is reduced to consideration of the dynamics of an ensemble of interacting oscillator modes, part of which draw energy from the source, while part give it to a thermostat (dissipate it). Here we consider basic models of this type.

In media with nonlinearity quadratic in the field, the elementary interaction is the three-wave interaction (synchronism condition $\omega_1 - \omega_2 - \omega_3 - \delta = 0$, $\mathbf{K}_1 - \mathbf{K}_2 - \mathbf{K}_3 = 0$):

$$\begin{aligned} \dot{a} &= \sigma_1(a_j) - a_2a_3e^{-i\delta t} \\ \dot{a}_2 &= \sigma_2(a_j) + a_1a_3^*e^{i\delta t} \\ \dot{a}_3 &= \sigma_3(a_j) + a_1a_2^*e^{i\delta t} \end{aligned} \tag{2.18}$$

Here the a_i are the complex wave amplitudes, which are assumed to be spatially homogeneous (the normalization is chosen so that the interaction coefficients are equal to unity), the σ_i are linear terms describing the pumping of energy and dissipation, and δ is the detuning from exact synchronism.

1. The character of the energy exchange between the unstable wave ω_1 and the damping pair ω_2 and ω_3, i.e., when $\sigma_1 = \gamma_1a_1$, $\sigma_{2,3} = -\nu_{2,3}a_{2,3}$ (flow of energy downward in the spectrum) depends essentially on the ratio of γ_1 and $\nu_{2,3}$. In the work of Vyshkind (1978) using numerical analysis, she found a chaotic exchange of energy between modes over a rather wide range of parameters. The chaos arose as the result of a chain of successive doubling bifurcations. A

simple investigation of the resulting strange attractor is difficult, since the system of differential equations following from (2.15) in this case is of order 4. More promising in this respect is the study of the degenerate case $\nu_2 = \nu_3$, since the amplitudes of low frequency waves become equal for $t \to \infty$, and the system (2.15) can be represented in the form

$$\dot{X} = z - 2Y^2 + \bar{\delta}Y + \gamma X$$
$$\dot{Y} = 2XY - \bar{\delta}X + \gamma Y \qquad (2.19)$$
$$\dot{Z} = -2Z(X+1)$$

Here $X = (|a_1|/\nu)\sin\psi$, $Y = (|a_1|/\nu)\cos\psi$, $Z = (|a_{2,3}|^2/\nu)$; $\psi = \arg a_1 - 2\arg a_{2,3} - \delta$, $\gamma = \gamma_1/\nu$, $\bar{\delta} = \delta/\nu$. At exact synchronism ($\delta = 0$) and $\gamma < 2$, all the trajectories in the phase space tend toward the planes $Z = 0$ or $Y = 0$ when $t \to \infty$,⁺ on which there are no stable equilibrium states or limit cycles. Stabilization of the unstable mode by energy transfer to equal low-frequency modes is consequently impossible in this case. Stabilization is possible for nonzero, even very small, detuning. The energy exchange can then be either static (stable stationary point) or periodic (limit cycle) (Vyshkind, 1977), or chaotic (strange attractor) (Wersinger, Finn and Ott, 1980).

For waves close to exact linear synchronism one can in principle have a nonlinear detuning, associated with the dependence of the phase velocity of the waves on their energies. In Eq. (2.15) there corresponds to this the inclusion of terms of the type $ia|a|^2$ (Dubrovin, Kogan and Rabinovich, 1978). Instead of (2.16) they studied the model

$$\dot{X} = z - 2Y^2 + (\delta - \alpha Z)Y + \gamma X$$
$$\dot{Y} = 2XY - (\delta - \alpha Z)X + \gamma Y \qquad (2.20)$$
$$\dot{Z} = -2Z(X+1)$$

Figure 2.13 shows the phase portrait of a typical attractor of this system. In this case also the most common way in which strange attractors arise is by successive bifurcation doubling of cycles.

2. If the sources of energy of the interacting waves are external high frequency fields, then a sufficiently general model is that of Pikovskii, Rabinovich and Trakhtenberg (1978) when there is linear

⁺ This follows from the fact that the function $P = ZY$ satisfies the equation $P = (\gamma - 2)P$ (Vyshkind, Rabinovich and Fabrikant, 1977).

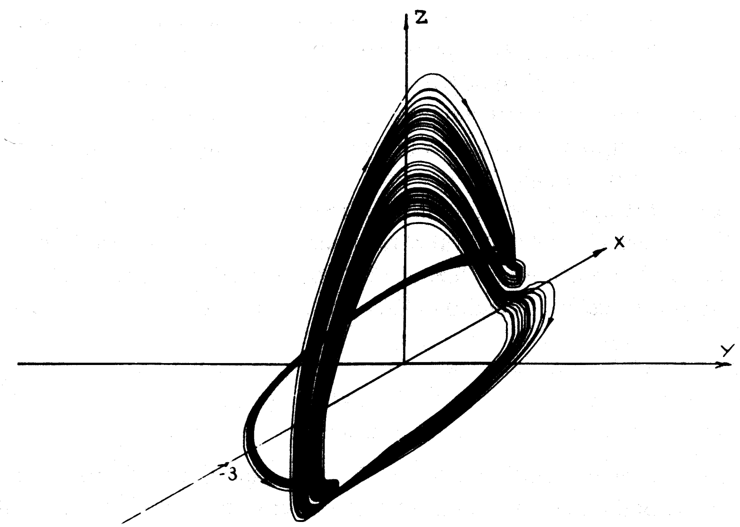


Figure 2.13. Phase portrait of typical attractor for the system (2.20).

damping in all three waves, while the supply of energy comes through a feeder wave ω_0, resonant with $\omega_{1,2}$; $\omega_0 = \omega_1 + \omega_2$, $\mathbf{K}_0 = \mathbf{K}_1 + \mathbf{K}_2$, whose amplitude is assumed fixed. In this case, $\sigma_1 = ha_2^* - \nu_1 a_1$, $\sigma_2 = ha_1^* - \nu_2 a_2$, $\sigma_3 = -\nu_3 a_3$, where h is proportional to the pumping amplitude. The peculiarity of this problem when $\delta = 0$ is that all directions in the six-dimensional phase space corresponding to a change of phase are stable, i.e., all amplitudes can be taken real on the attractor. We shall discuss this situation somewhat more in detail, since it allows us to cross the bridge to models of hydrodynamic turbulence (and, in particular, to the Lorenz model). The Eqs. (2.15) reduce to the following system for the real variables $x = |a_2|$, $y = |a_1|$, $z = |a_3|$:

$$\dot{x} = hy - \nu_2 x + yz$$
$$\dot{y} = hx - \nu_1 y - xz \qquad (2.21)$$
$$\dot{z} = -\nu_3 z + xy$$

It is easily noted that the system (2.18) is very similar to the Lorenz system

$$\dot{x} = \sigma y - \sigma x$$
$$\dot{y} = rx - y - xz \qquad (2.22)$$
$$\dot{z} = -bz + xy$$

differing from it in only a single term. As a numerical analysis showed, this difference is unimportant, and the strange attractor in (2.18) is completely analogous to the Lorenz attractor. Using this analogy, one can interpret the Lorenz attractor as a system of nonlinear coupled modes, two of which (including decaying ones) are excited "parametrically."

Just as the system of three interacting waves (2.22) can be regarded as the "basic model" of wave turbulence, so the construction of a model of hydrodynamic turbulence can start from the elementary model proposed by Obukhov (1974) of a triplet of interacting modes

$$\begin{aligned} \dot{u}_1 &= -u_2 u_3 \\ \dot{u}_2 &= u_1 u_3 \\ \dot{u}_3 &= u_1 u_2 \end{aligned} \tag{2.23}$$

These equations, which coincide with the equations of motion of a free rigid body, are the simplest finite-dimensional analog of the Euler equations of motion of a viscous incompressible liquid. Combining several such triplets, one can construct a system modeling the cascade transfer of energy from large-scale modes to fine-scale ones.

In such a multiscale system a strange attractor was found by Sonechkin (1979); a similar problem was also solved by Boldrighini and Franceschini (1979). We note that the type of nonlinearity in the basic model (2.20) coincides with the type of nonlinearity in the Lorenz system (2.19)—a quadratic conservative nonlinearity. But in the Lorenz system the main role is played by the linear terms, of which there are none in (2.20). The point is that (2.20) is a model of the dynamics of a homogeneous ideal liquid, while the Lorenz system is obtained by simplifying the Boussinesq equations, describing the motion of a heated inhomogeneous liquid. A general discussion of model methods for describing thermal convection using the simplest finite-dimensional systems was given by Dolzhanskii and Pleshanova (1979) and Glukhovskii and Dolzhanskii (1980). In these papers it was shown that the simple models of convection consist of interacting Lorentz triplets. In these models, for certain parameters one finds stochastic behavior, which, however, differs from the Lorenz attractor (cf. also Curry, 1978). A similar approach to wave turbulence was discussed in the paper of Astashkina and Mikhailov (1980). They considered the following cascade process of excitation of turbulence of spin waves in an antiferromagnetic crystal. The source of energy is a uniform ($\mathbf{K}_0 = 0$) pump with frequency ω_0. It excites a pair of waves $\omega_0 = \omega_1 + \omega_2$, $O = \mathbf{K}_1 + \mathbf{K}_2$. Each of the waves of the excited pair in turn transfers the energy to another pair $\omega_{1,2} = \omega^+_{1,2} + \omega^-_{1,2}$, $\mathbf{K}_{1,2} = \mathbf{K}^+_{1,2} + \mathbf{K}^-_{1,2}$. As a result the problem reduces to the investigation of a system of ordinary differential equations of fourth order, in which a strange attractor was found by Astashkina and Mikhailov.

The models treated above for the appearance of stochasticity in resonant interacting waves (modes) are possible only when the wave spectrum is decaying and can be supplemented by the condition of three-wave synchronism. If decay processes are forbidden then stabilization of the instability will be achieved by transfer of energy to neighboring (near to the unstable one) scales. The simplest model of this kind can be gotten by generalizing the familiar model of Landau, by including a spreading of the spectrum in the process of self-modulation of the wave packet, i.e., excitation of neighboring modes. The starting point can be a nonlinear parabolic equation with complex coefficients

$$a_t = \gamma_0 a + (\beta + i\kappa) a_{xx} + (i\alpha - \rho) a |a|^2 \tag{2.24}$$

where the complex amplitude of the wave is $a(x,t)$. Eq. (2.24) describes the behavior of excitations in nonequilibrium dissipative media near the threshold of instability, when the spectrum of unstable excitations is narrow, and their increment is small.

In the work of Rabinovich and Fabrikant (1979) Eq. (2.24) was studied on a simple three-mode model, including an unstable mode k_0 and a damping pair of satellites $k_{1,2} = k_0 \pm \Delta k$. As a result they obtained the following system of equations:

$$\begin{aligned} \dot{x} &= y(z - 1 + x^2) + \gamma x \\ \dot{y} &= x(3z + 1 - x^2) + \gamma y \\ \dot{z} &= -2z(\nu + xy) \end{aligned} \tag{2.25}$$

analogous to the system (2.20). A detailed numerical study of these equations enabled them to construct bifurcation diagrams on the parameter plane (Fig. 2.14). The appearance of the strange attractor is shown in Fig. 2.15. The fact that the simplest model of Eqs. (2.24) has a strange attractor enables one to explain the occurrence of turbulence in the full equations (2.21), as was established by Kuramoto and Yamada (1976) by a direct numerical integration.

Naturally the simplest systems cannot pretend to give a quantitative description of developed wave or hydrodynamic turbulence. But

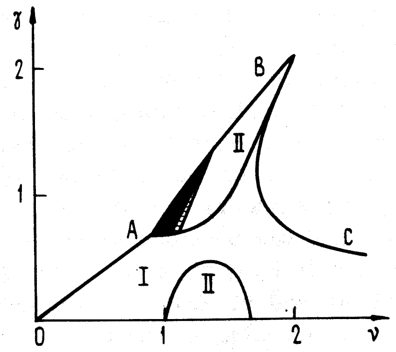


Figure 2.14. Division of the plane of parameters (γ, ν) of the system (2.20) into regions with different dynamics.
I. unmodulated regime.
II. simple periodic modulation.
dashed region—doubling of period of modulation.
dark wedge—stochastic modulation.
No stabilization above the curve ABC.

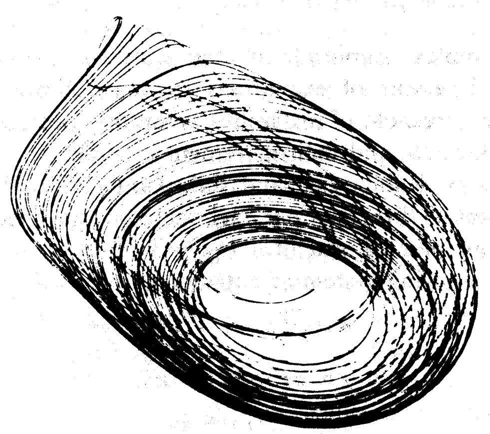


Figure 2.15. Strange attractor in the system (2.25) for $\gamma = 0.9$, $\nu = 1.1$.

they can clarify the mechanism of its appearance. In particular, a favorable factor for the appearance of stochasticity is instability of high frequency waves and transfer of their energy to low frequency ones.

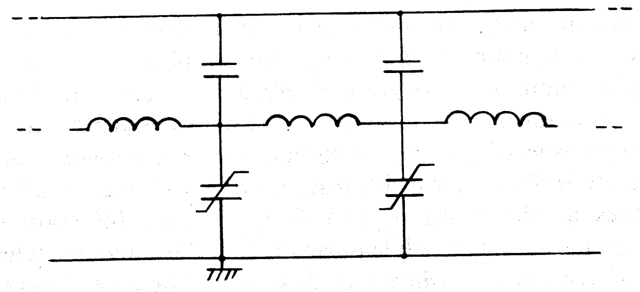
III. Complex Systems

As we have seen, even in simple systems almost always the problem arises of separating the true stochasticity determined by the dynamics

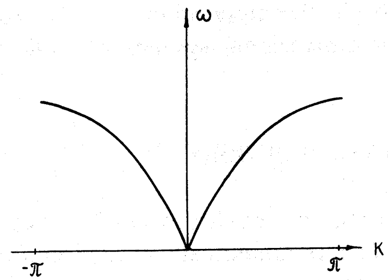
of the system from the stochasticity that owes its origin to the presence of external or internal noise. This problem is especially acute in complex (number of degrees of freedom > 10) and distributed systems. Basically the final answer to the question can be given only by a comparison of the theory (within whose framework such true stochasticity is discovered) with real experiments. Since true stochasticity arises as the result of completely definite bifurcations that complicate the spectrum, while during the establishing of stochasticity individual realizations, when t is close to 0 evanesce exponentially with increasing t, these are points to which one pays attention in experiments. We begin the description of such experiments with the simplest ones—one-dimensional systems of radio physics and electronics.

1. Stochastic Modulation of Waves

Stochastic modulation of one-dimensional waves was observed by Ezerskii et al. (1979) in parametrically excited uniform pumping of an *LC* network. Its dispersion curve and equivalent circuit are shown in Fig. 3.1. The nonlinearity of such a "medium" is determined by the dependence $Q(u)$ (cf. Fig. 3.1), where $Q(u) = c_0(u + \alpha_1 u^2 + \alpha_2 u^3 + \cdots)$. With increasing amplitude of the pumping harmonic in the line, matched at the ends, one observed the following evolution of the spectrum of oscillations (cf. Fig. 3.2). When one exceeds the threshold of parametric instability $u_p > u_{p_1}$, one excites a pair of oppositely directed waves with wave vectors of equal magnitude, $\mathbf{K}_{1,2} - \mathbf{K}_1$. To this corresponds the establishment in the network of a spatially homogeneous (in amplitude) regime of oscillations at frequency $\omega_p/2$. Such a regime exists until $u_p \leqslant u_{p_2}$. The next transition (when $u_p = u_{p_2}$) corresponds to the excitation of several spatial modes with closely equal wave numbers $k = k_1 + \Delta k$. These modes with almost equal eigenfrequencies do not appear in the frequency spectrum because of mutual frequency locking. A further increase of pumping leads to the replacement of the single frequency regime of locking (i.e., a regime with complete synchronization) by a regime of partial synchronization $u_p \geqslant u_{p_3}$, characterized by the presence of a pair of symmetrically located satellites around $\omega_p/2$: $\omega = \omega_p/2 + \Omega$. Further increase of amplitude of pumping leads to the appearance of subsatellites $\omega = \omega_p/2 \pm n(\Omega/4)(u_p = u_{p_4})$, and finally, when $u_p = u_{p_{cr}}$ the stochastic regime, characterized by a continuous spectrum, appears. The transition to stochasticity via a succession of appearances



a.



b.

Figure 3.1. One-dimensional LC-network, parametrically excited by uniform pumping.
a. equivalent circuit.
b. dispersion characteristic.

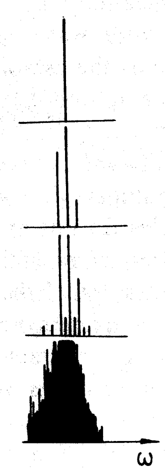


Figure 3.2. Evolution of oscillation spectrum in a parametrically excited network with increasing amplitude of pumping.

of subharmonics shows that in this system there is realized the mechanism for the development of strange attractors, common to many dissipative systems, which reduce to a one-dimensional mapping. The fact that turbulence in such a dissipative system is "intrinsic" and not the result of amplification of fluctuations is also confirmed as follows. It was found that the introduction into the line of a spatially uniform noise field from an auxiliary source with spectral intensity of the noise down to $I_\omega/I_0 \sim 5 \times 10^{-3}$* has no qualitative effect on the character of the transitions and the spectrum of parametric turbulence. The noise only reduces insignificantly the threshold for onset of the stochastic regime.

A regime of stochastic modulation can also appear in an autonomous wave system as the result of development of intrinsic instability (for example, beam instability in a plasma). B. P. Bezruchko, S. P. Kuznetsov and D. I. Trubetzkov (1979) observed transitions, analogous to those just discussed, to a regime of vibrations with stochastic modulation in a distributed electron generator in the short wave region—a backward-wave tube. A block diagram of this generator is shown in Fig. 3.3. The electron beam moves through a retarding system along which waves with a transverse electric field propagate. The parameters of the system are selected so that the phase velocity of this wave at some frequency Ω coincides with the beam velocity $v_{ph}(\Omega) \approx v_0$ while the group velocity points opposite to it. The output signal is taken from the same end of the retarding system to which the beam goes. Then through interaction of the wave perturbations of frequency $\omega \approx \Omega$ with the electron beam one gets a distributed inverse feedback and an absolute instability appears, leading to a stationary regime of generation. The character of this regime is determined by a single parameter: $\mathfrak{L} = \beta l (Ik/4u)^{1/3}$, where β is the wave number of the wave that is synchronous with the current, l is the interaction length, I is the constant component of the beam current, u is the accelerating field, and k is a parameter of the system with the dimensions of a resistance. The succession of bifurcations observed in this system along its path to the regime of stochastic modulation (as the parameter $\mathfrak{L}$ increases) is shown in Fig. 3.4. When $\mathfrak{L} \gtrsim \mathfrak{L}_{cr}$ a stochastic regime begins, characterized by a continuous spectrum.

In the experiments described, using the BWO (backward wave oscillator) the parameters of the retarder system were changed as well

*Here I_0 is the total intensity of the excitations in the regime of parametric turbulence, I_ω is the intensity of the external noise in the range $0 < \omega < \omega_0$.

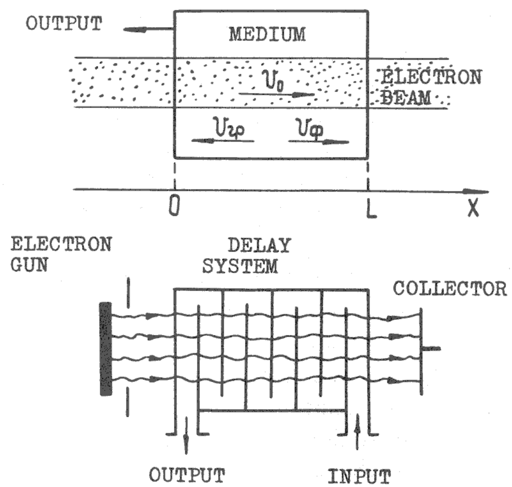


Figure 3.3. Block diagram of distributed autooscillator "backward wave oscillator" (BWO).

as the electron beam, the feed, etc, and it was found that the character of the transitions along the path to chaotic modulation did not change qualitatively, and in all variants of the experiment was determined by the same dimensionless parameter $\mathcal{L}$. Such similarity obviously proves the unimportance basically of fluctuations (in particular, noise in the electron beam) in the development of the stochastic regime in the BWO. The regime of stochastic self-oscillations could be destroyed by using an external synchronizing signal. The most effective synchronization occurred if a periodic action was applied at frequencies corresponding to the left satellites in the spectrum of the preturbulent regime. One also observed the reverse process—in the action of periodic signals on the BWO in the preturbulent regime, the discrete spectrum corresponding to periodic modulation was replaced by a continuous spectrum if there was sufficiently large detuning between the frequency of the applied signal and that of the satellite. All these changes occurred at the same beam current (i.e., with the same fluctuations in the electron beam), which once again proves the dynamical origin of the observed chaotic regime.

2. *Stochasticity of Vortices in a Viscous Fluid*

We know that developed hydrodynamic turbulence can only be described by a very complex dynamical system—a system containing

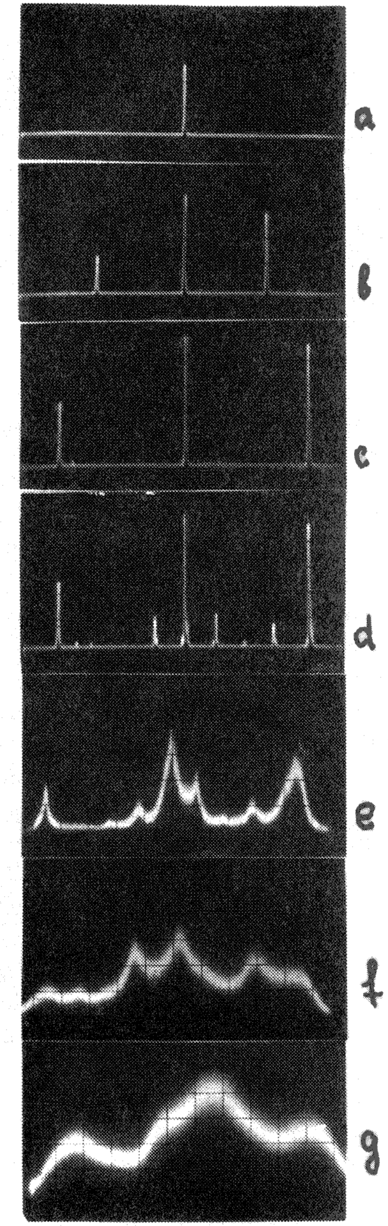
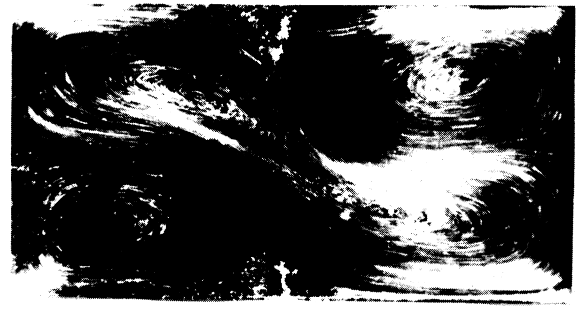


Figure 3.4. Power spectrum of output signal of distributed electron generator (BWO) for u = const and different beam currents $\sim\mathcal{L}$ (the abscissa gives normalized frequency detuning).
a—single frequency generation.
b, c, d—regime of periodic modulation.
e, f, g—stochastic automodulation.

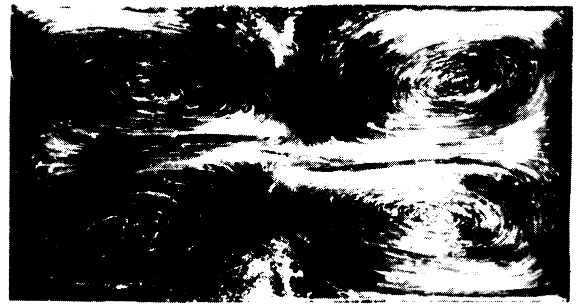
a large number of degrees of freedom. But for many hydrodynamic flows, very close to the point of transition to turbulence (for Re $\gtrsim Re_{cr}$) a characteristic feature is the appearance of a not too large number of elementary excitations (modes). Such a situation is most natural for various flows in cavities (resonators) as a result of artificial depletion of the spectrum of the modes of the flow. Simplest in this sense is the convective motion of a viscous liquid in a closed loop (Creveling et al. 1975). In such a toroidal resonator the flow is actually uniform, and stochasticity manifests itself only in the random change of directions of rotation of the fluid in time. Next in simplicity is probably the appearance of turbulence in a vertically standing "matchbox"—a Haley-Shaw cell, first studied by Lyubimov, Putin and Chernatinskii (1977). They observed chaotic convection in a parallelepiped, whose dimensions are clear from Fig. 3.5. On the wide rigid walls of the cell they maintained a constant temperature gradient, independent of the character of the convection in the cell. In a series of experiments one of the wide side walls of the cell was transparent, which permitted observation of transitions of some motions to others with increasing Rayleigh number Ra. When $Ra > Ra_1$ single-vortex convection was established in the cavity; this was replaced by two- or four-vortex motions when $Ra \geqslant Ra_2$; for values $Ra_2 < Ra < Ra_3$ there was a dynamical oscillatory regime. When $Ra > Ra_3$ irregular convection appeared. The pictures of the flow shown in Figs. 3.5a, b, c followed each other randomly in time. The pulsations of temperature inside the cell corresponding to such a stochastic regime (measured using a thermocouple) are shown in Fig. 3.6.

In this case the appearance of stochastic motion in the resonator cell is related to a global reorganization of the whole flow picture (cf. below). Chaos arises differently in a cylindrical layer under rotation of the inner cylinder (Couette flow between cylinders).

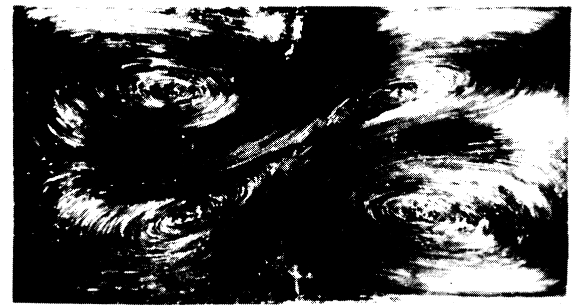
Such a flow was studied in detail by L'vov and Predtechenskii (1979) in a geometry differing from the analogous setup of Fenstermacher, Swinney and Gollub (1978) essentially in having a larger gap between the cylinders. The outer and inner cylinders were 300 mm high, radii 35 mm and 55 mm, with the gap between them filled with water. By stabilization of the temperature and the rotation period, the Reynolds number Re of the flow was held fixed to 0.01%. The flow rate was measured with a laser-Doppler anemometer with data processing on a digital computer. Figure 3.7 shows a series of power



a.



b.



c.

Figure 3.5. Turbulent convection in a Haley-Shaw cell; a, b, c are different types of motion for the same value of $Ra > Ra_3$, randomly interchanging in time.

spectra of the flow for different Re (scale is log base 10). The vertical bands indicate the frequency of rotation of the driver and its harmonics Ω, 2Ω, 3Ω. The upper spectrum (Re = 1020) corresponds to a flow in which azimuthal waves are excited on the background of the Taylor vortex (the boundaries of the vortex are sinuous). Because of the rotation of the vortex the fluid velocity at a fixed point depends periodically on the time. In the power spectrum this dependence is

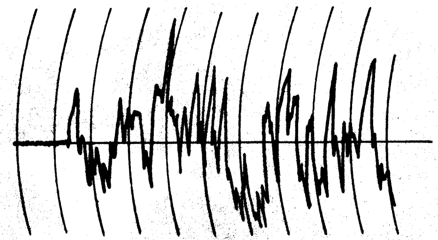


Figure 3.6. Time dependence of temperature differences measured with a thermocouple with junction placed horizontally inside a Haley-Shaw cell.

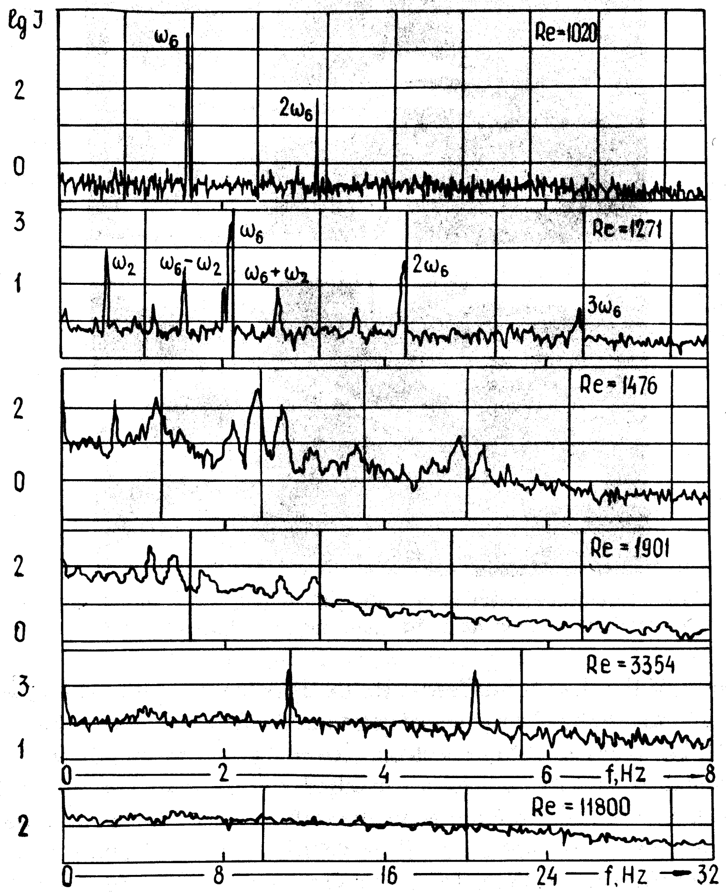


Figure 3.7. Evolution of power spectrum of Couette flow between cylinders with increasing Re.

shown by the peak ω_6 (the sixth azimuthal mode is excited), three orders of magnitude greater than the noise level. The second harmonic $2\omega_6$ is visible on the right. With increasing rate of rotation (Re) the spectrum becomes successively more complex, and for Re $\simeq$ 1270 a broadening of the peaks in the power spectrum appears, corresponding to chaotization of the flow. As Re grows these peaks continue to widen and finally the spectrum becomes almost continuous. Details of the evolution of the power spectrum in the interval Re = 1000—1200 are shown in Fig. 3.8. The next figure shows oscillograms of the azimuthal waves, obtained by filtering the signal and subsequent detection. The realizations in this Fig. (3.9) a, b, c, d, e correspond precisely to the spectra a, b, c, d, e in Fig. 3.8. Initially the azimuthal waves are purely harmonic (cf. Fig. 3.9a), then a sinusoidal modulation appears which with further growth is replaced by chaotic modulation. For larger values of Re, as a result of synchronization of

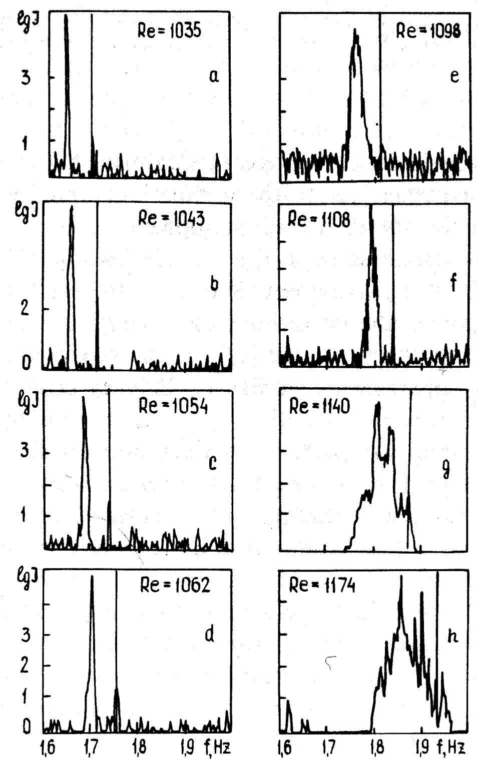


Figure 3.8. Evolution of power spectrum of Couette flow in the interval 1000 < Re < 1200.

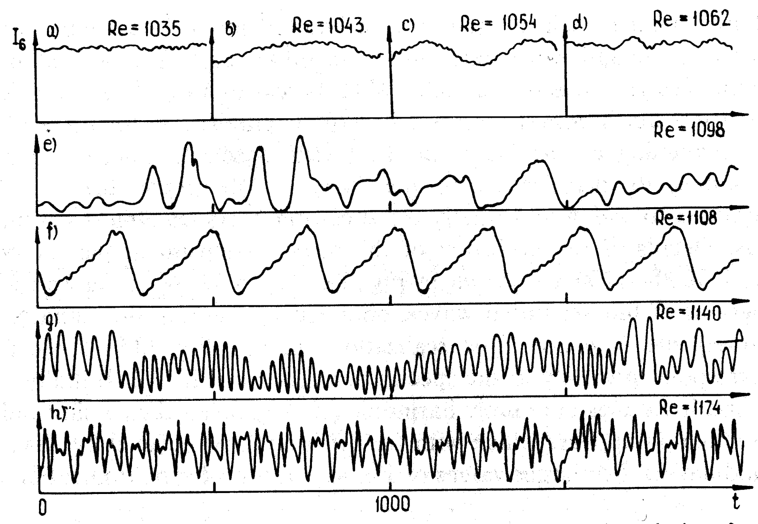


Figure 3.9. Oscillograms of time dependence of modulation of the velocity of azimuthal waves for $1000 < Re < 1200$. Realizations a, b, c, d, e precisely correspond to the spectra shown in the preceding figure.

the modes a regular regime is again established, but it is no longer sinusoidal, but rather, sawtooth modulation, and finally, when Re = 1140 stochastic modulation again appears.

We shall postpone temporarily the discussion of the nature of stochasticity in this experiment. We now present the results of an experiment studying the transition to turbulence in Couette flow for a more complex geometry (spherical). Such transitions in a flow between rotating spheres were studied by Belyaev et al. (1978). In their setup the gap between spheres $\delta = (r_2 - r_1)/r_1 = 1.006$; only the inner sphere was rotated, so that the Reynolds number $Re = \rho r_1^2/\nu$ varied over a wide range. The working fluid was a solution of oil in gasoline. The thermoanemometer readings were analyzed on a computer, and the energy spectra of pulsations and the autocorrelation function were calculated.

A feature of spherical Couette flow (as compared to cylindrical) is that the Taylor vortices that appear are not identical in form from the equator to the pole. As a result, in the linear approximation, the frequencies of azimuthal waves at different vortices depend strongly on the vortex number.

The succession of transitions to the stochastic regime observed by

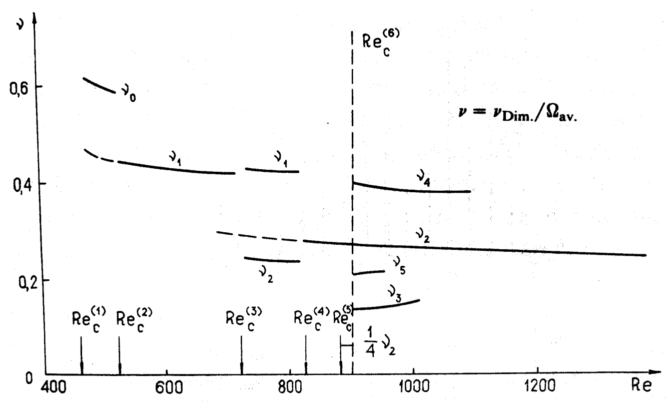


Figure 3.10. Diagram illustrating the sequence of transitions to turbulence in spherical Couette flow.

Belyaev et al. with increasing Reynolds number are shown in Fig. 3.10. In the laminar regime one observes a quite sparse spectrum of azimuthal oscillations. This is related to a complete synchronization of the oscillations of the vortices at the equator and at the pole. The onset of turbulence is preceded by the following sequence of bifurcations. At $Re = 460 \pm 10$ an unstability appears at frequency ν_0, corresponding to a four-vortex regime, at $Re_2 = 525$ the peak at ν_0 disappears and a frequency ν_1 appears corresponding to a three-vortex regime, then (at $Re_3 = 720$) a two-frequency quasiperiodic regime begins, and at $Re_4 = 820$, the peak at ν_1 drops out and a regime appears with the frequency ν_2, corresponding to two vortices at $Re_5 = 880$ a subharmonic at $\nu_{2/4}$ appears with its harmonics, vibrations at the pole. At $Re_6 = 895$ a regime of amplitude modulation of the vortex sets in, and satellites at the frequency $\nu_{2/4}$ and its odd harmonics appear in the spectrum. Obviously there is then a desynchronization of the oscillations at the equator and the pole, which is related to the appearance of the satellites. The further dependence of the frequency of oscillations at the pole ν_n and at the equator ν_2 on Reynolds number is different, i.e., these frequencies are actually incommensurable (unsynchronized). Then at $Re = 902$, the frequency $\nu_{2/4}$ disappears and frequencies $\nu_3 = 0.1363$ and $\nu_5 = 0.2078$ appear. At $Re = 915$ the subharmonics vanish and we get islands of continuous spectrum. A significant complication of the spectrum and damping of the autocorrelation function of the flow begins at $Re > 930$ (cf. Fig. 3.11).

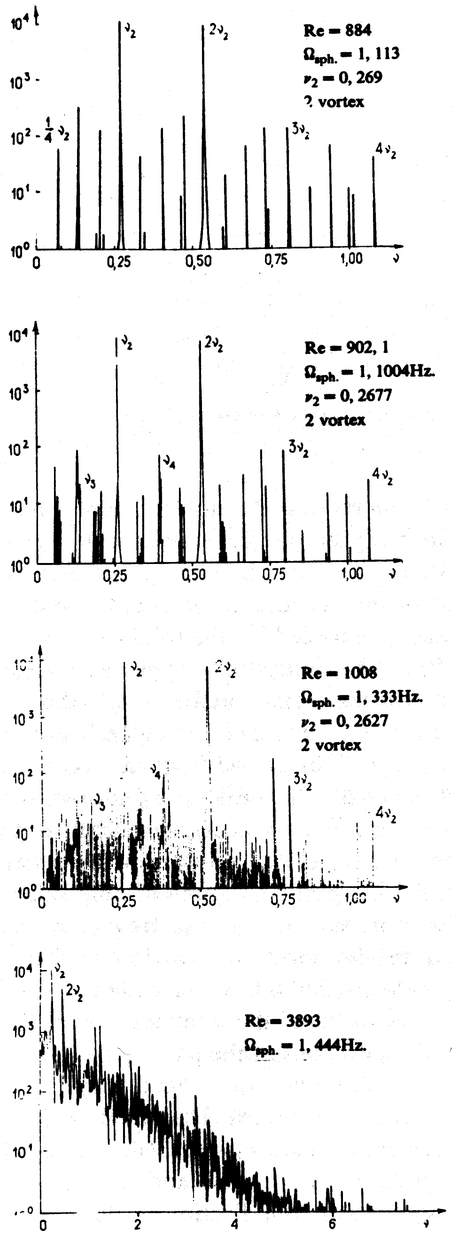


Figure 3.11. Evolution of power spectrum of fluid flow between two spheres with increasing Re.

Thus the transition to turbulence in spherical Couette flow is fundamentally more complicated than the corresponding flow between cylinders. This is apparently explained by an essentially more complicated spectrum of elementary excitations in the spherical geometry. The appearance of islands of continuous spectrum, in particular, may be related to the partial desynchronization of the modes at neighboring vortices, since the frequencies of azimuthal modes at neighboring vortices are, in general, incommensurable.

3. *Mathematical Description*

In contrast to the case of simple systems (for which the mathematical description of chaos can be obtained usually from the original equations of the problem), for complex systems such a procedure is not very promising. For example, even for the simplest hydrodynamic flow—convection in a layer, the description of the development of turbulence within the framework of the Boussinesq equation requires powerful computational methods and a great deal of machine time. Moreover, solution on a computer of the original equations, i.e., replacement of the physical experiment by a numerical one, is not so useful in constructing a physical theory as obtaining models that are simpler and more accessible to analysis by using various idealizations. The consideration of such models for the experiments discussed above is done by increasing the "crudeness" of the assumptions made in deriving them.

In the electron beam (BWO) system, the stochastic modulation can be described in detail within the framework of the averaged equations obtained from the equations for the beam and the field (Ginzburg, Kuznetsov and Fedoseev, 1978). As already mentioned, the effective interaction of the beam with the backward-wave field can occur if any one of the spatial harmonics has a velocity close to the velocity of the electrons. Then if we write the field of this harmonic in the form $E(x,t) = \mathrm{Re}\{\mathcal{E}(x,t)\cdot \exp[i\Omega(t - x/v)]\}$, where the frequency Ω is determined from the synchronism condition $v_{ph}(\Omega) = v_0$ (where v_0 is the beam velocity) one can then obtain the equation for the normalized slowly varying amplitude $F \sim \mathcal{E}$:

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial x} = -\frac{\mathfrak{L}}{\pi}\int_0^{2\pi} e^{-i\theta(\alpha)}\,d\alpha$$
$$\frac{\partial^2\theta}{\partial x^2} = -\mathfrak{L}^2\,\mathrm{Re}\left(Fe^{i\theta(\alpha)}\right) \tag{3.1}$$

with boundary and initial conditions

$$\theta(\alpha)|_{x=0}=\alpha, \qquad \frac{\partial\theta}{\partial x}\bigg|_{x=0}=0, \qquad F|_{x=l}=0$$
$$F|_{t=0}=F_0(x) \tag{3.2}$$

The character of the developing self-oscillations, as the experiments show, depends only on the single parameter $\mathfrak{L}=(Ql/v_0)(Ik/4u)^{1/3}$, which in this case plays the role of the Reynolds number.

It follows from the analysis of the system (3.1), (3.2) (Bezruchko, Kuznetsov and Trubetskov, 1980) that for $2.0\lesssim\mathfrak{L}\lesssim2.9$ an unmodulated regime develops; when $\mathfrak{L}\gtrsim2.9$ a periodic modulation appears, after which the process becomes aperiodic. Statistical treatment of the realizations shows that the calculated power spectrum in the regime of stochastic modulation agrees with that observed experimentally; the autocorrelation function falls off sufficiently fast. The agreement of the results of experiment and theory confirms the dynamical origin of chaos in the experiment using the BWO.

In this example the model, although simplified (by the averaging over high frequency oscillations) when compared to the original equations still preserves their main feature—an infinite number of degrees of freedom. It is true that in the numerical computation this system is also replaced by a finite dimensional one, but with a quite large number of degrees of freedom. The number of degrees of freedom that must be taken into account in constructing a model obviously depends on the problem for which the model is built. If it is necessary, for example, to establish the possibility in principle of having chaos of self-modulation then, as we have seen in the preceding paragraph, three modes are sufficient, but if we want to describe in detail all the transitions observed with increasing supercriticality along the way to chaos, and the evolution of the spectrum of the stochastic regime, then the dimensionality of the finite-dimensional model must increase.

Let us illustrate this on the example discussed above of turbulent convection in a Haley-Shaw cell (Lyubimov, Putin and Chernatinskii, 1979).

Assuming that the structure of the flow along the depth of the cell (coordinate z) is given, we can introduce the stream function $\psi: v_x=-\partial\psi/\partial y$, $v_y=\partial\psi/\partial x$. Then in dimensionless units the equations for the vorticity $\omega=-\psi_{xx}-\psi_{yy}$ and the deviation T of the temperature from equilibrium take the form (they are gotten from the equations of free convection in the Boussinesq approximation):

$$\begin{aligned} P_r^{-1}\left[\omega_t+J(\psi,\omega)\right]&=\Delta\omega+RaTx\\ T_t+J(\psi,T)&=\Delta T+\psi x \end{aligned} \tag{3.3}$$

P_r is the Prandtl number, Ra is the Rayleigh number, and J is the Jacobian. Looking for a solution of (3.3) in the form of an expansion in modes

$$\psi(x,y,z)=\sum\psi_{nm}(t)\sin\frac{n\pi}{L}x\sin\frac{m\pi}{H}y\cos\frac{\pi}{2}z$$
$$T(x,y,z)=\sum T_{nm}(t)\cos\frac{n\pi}{L}x\sin\frac{m\pi}{H}y\cos\frac{\pi}{2}z$$

where H and L are the vertical and horizontal dimensions of the cell, it is not difficult to obtain for $\psi_{nm}(t)$, $T_{nm}(t)$ a finite dimensional system of the form $(a\sim\psi_{n,m},T_{n,m})$

$$\dot a_k=\sum\sigma_k^{lm}a_la_m+\sum\gamma_k^la_l \tag{3.4}$$

As we saw, for $Ra>Ra_0$ a one-vortex convection appears in the cell —the modes ψ_{11}, T_{11}, T_{02} are different from zero, while other modes are not excited. The system of equations (3.4) for the excited modes coincides with the well known Lorenz system. In principle this model could already describe the appearance of stochastic pulsations, but the stochasticity described by the Lorenz system cannot be achieved —at very small Rayleigh numbers new modes are excited. Using the information from the experiment that after the one-vortex regime a regime with four vortices appears, we also include the modes ψ_{22}, T_{22}, ψ_{31}, T_{31}, T_{04}. Then we get for them a system of equations of the form (coefficients of order unity are omitted):

$$\begin{aligned}
\dot\psi_{11}&=T_{11}+\psi_{31}\psi_{22}\\
\dot T_{11}&=\psi_{11}-T_{11}-\psi_{11}T_{02}+T_{31}\psi_{22}+T_{22}\psi_{31}\\
\dot T_{02}&=-T_{02}-\psi_{31}T_{31}-T_{02}+\psi_{11}T_{11}\\
\dot\psi_{31}&=T_{31}-\psi_{31}+\psi_{11}\psi_{22}\\
\dot T_{31}&=\psi_{31}-T_{31}+\psi_{31}T_{02}-T_{11}\psi_{22}+T_{22}\psi_{11}\\
\dot T_{04}&=-T_{04}+\psi_{22}T_{22}\\
\dot\psi_{22}&=T_{22}-\psi_{22}-\psi_{11}\psi_{31}\\
\dot T_{22}&=\psi_{22}-T_{22}-\psi_{22}T_{04}-T_{11}\psi_{31}-T_{31}\psi_{11}
\end{aligned} \tag{3.5}$$

This system contains three coupled systems of the Lorenz type. In the numerical investigation they found precisely the same transitions along the path to the stochastic regime that were observed experimentally by Lyubimov, Putin and Chernatinskii—an unmodulated regime was replaced by a periodically modulated one, and then stochastic convection appeared. Thus a comparatively low-mode model appeared to be fully applicable for convection in a "cell."

When our problem is to study the complex dynamics arising as the result of the development of secondary instabilities on a background of, for example, a periodic motion, the problem of constructing mode models directly from the original equations becomes extremely complicated. Here already the model itself must be constructed using a computer (cf. Gertsenshtein and Schmidt, 1977). The development of any sort of qualitative pictures and the construction of a theory on a physical level is thus naturally made extremely difficult. In such situations, phenomenological models, based on some physical pictures and on experiment, are extremely useful. Just such a model was constructed by L'vov and Predtechenskii (1979) to describe the onset of chaotic modulation of the azimuthal waves appearing on a background of stationary flow—Taylor vortices in a "cylindrical Couette" (cf. above).

L'vov and Predtechenskii wrote down the model equations directly for $a_n(t)$—the amplitudes of bending of the boundary between vortices in a pair with number n:

$$\frac{da_n}{dt} = \gamma a_n + (i\eta - \rho)|a_n|^2 a_n + \frac{\alpha + i\beta}{4}(a_{n+1} + a_{n-1} - 2a_n) \quad (3.6)$$

The first two terms on the right of this equation coincide with the right side of the well known Landau equation (1941). If $\gamma > 0$ this equation describes the growth and stabilization of the bending oscillations of vortices because of self-interaction and interaction with other parts of the flow. The model (3.6) also includes interaction between vortices. Since it follows from experiment that this interaction is small, we can here limit ourselves to terms of first order in the amplitude. The coefficients of this model can, in principle, be determined directly from experiment. Equation (3.6) is essentially the differential-difference analog of the nonlinear Schroedinger equation for nonequilibrium media (cf (2.25)).

L'vov and Predtechenskii replaced the real flow by the system (3.6) with $n = 30$, the number of vortices in the experiment, and studied it

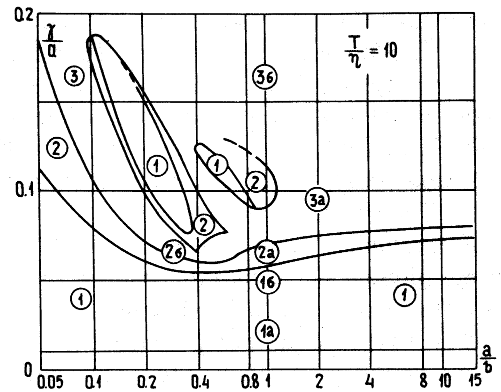


Figure 3.12. Division of space of parameters γ/α, α/β into regions with different character of flow in "cylindrical Couette."

in detail on a computer. Figure 3.12 shows the breakup of the space of parameters γ/α, α/β into regions with different flow character. Region I corresponds to unmodulated flow—stable states of equilibrium; II is flow with periodic modulation (limit cycles); III is stochastic modulation (a strange attractor in the phase space of (3.6)). For a suitable selection of the coefficients α and β ($\alpha/\beta = 0.8$), with increasing Reynolds number exactly the same sequence of rearrangements of the flow were observed as in the experiment (cf. above).

For small supercriticalities the system (3.6) of 30 equations can be truncated to three, and we go over to a model analogous to (2.25), in whose framework Rabinovich and Fabrikant also found chaotic modulation. Further increase of supercriticality leads from three equations to five, etc. In all cases a strange attractor corresponds to the turbulence observed in experiment, but the dimension of the model with which it is described increases in discrete steps as the Reynolds number increases.

Conclusion

The stochastic dynamics of dissipative systems, to which our survey is devoted, has entered in a natural way into the general theory of oscillations and waves, traditionally developed in Russia and then in the USSR. This branch of science attracts a wide circle of scientists—from "almost pure" mathematicians to specialists in specific areas (engineers, ecologists, meteorologists, etc.). The role of the theory of oscillations is significant, in particular, because of the development in it of international "oscillational" models and a language understandable in the same way by everybody interested in these problems.

Especially fruitful has been the approach used in the theory of oscillations at the first level of understanding this new domain, which we have tried to demonstrate in describing the stochastic dynamics of nonlinear systems. The point is that for this first stage of the problem the collecting of incomplete facts and phenomena from chemistry, radio engineering, physics, etc. and the construction of sufficiently general models is fundamental. The development of the investigations leads in a natural way to a deeper mathematical analysis of the elementary models, on the one hand, and to more detailed experiment (chemical, physical, etc.) requiring sophisticated measuring technique and data analysis on the other. We may, however, hope that such a differentiation will not lead to a breakdown of the close connection between mathematicians and physicists, which has now set it in the field of investigation of stochastic oscillations.

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