NONLINEAR RESONANT TWO-WAVE INTERACTION IN AN INHOMOGENEOUS MEDIUM

A.S. PIKOVSKY
Institute of Applied Physics, Academy of Sciences of the USSR, Gorky, USSR

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A one-dimensional steady-state nonlinear resonant two-wave interaction in a lossless periodically inhomogeneous medium is investigated. It is shown that the amplitudes and the phases of the waves vary in a stochastic manner.

Nonlinear resonant wave interactions play an important role in nonlinear optics, plasmas, hydrodynamics. In general, the amplitudes and phases of the coupled waves may vary either periodically or stochastically. The simplest example of regular energy exchange provides a steady-state process of two-wave interaction in a lossless homogeneous medium (with resonance conditions $\omega_2 = 2\omega_1, k_2 = 2k_1$). An exact solution of this problem is presented in ref. [1]. However, the addition of one more wave (the third harmonic $\omega_3 = 3\omega_1, k_3 = 3k_1$) radically changes the state of things: the corresponding system of ODEs has ergodic properties [2].

In the present paper it is shown that stochastic behavior of nonlinearly coupled waves may be caused by weak periodical inhomogeneities.

We describe the one-dimensional steady-state nonlinear interaction of two undamped waves in a periodically inhomogeneous medium by the equations

$$\mathrm{d}a_1/\mathrm{d}x = -a_1 a_2 \sin\theta, \quad \mathrm{d}a_2/\mathrm{d}x = a_1^2 \sin\theta,$$

$$\mathrm{d}\theta/\mathrm{d}x = a_2^{-1}(a_1^2 - 2a_2^2)\cos\theta + \Delta k(x). \tag{1}$$

Here $a_{1,2}$ are normalized real amplitudes of the waves (ω_1, k_1) and (ω_2, k_2); $\theta = \varphi_2 - 2\varphi_1$ is the phase difference and $\Delta k(x) = L_\mathrm{p}^{-1} \sin(L_\mathrm{i}^{-1}x)$ is the periodically varying wave number mismatch. The parameters L_i and L_p correspond to the period of the inhomogeneity and its intensity, respectively. We restrict our attention only to spatial dephasing effects, the variation of the remaining parameters is assumed to be small. Normalized wave amplitudes have the dimension of a reciprocal length, so the energy integral may be written in the form $a_1^2 + a_2^2 = \mathrm{const} = L_\mathrm{n}^{-2}$, where L_n is the characteristic length of energy exchange. Clearly, the parameters $L_\mathrm{n}, L_\mathrm{i}$ and L_p must be large in comparison with the wave length k_1^{-1}.

Thus our problem is the system of ODEs (1) subject to appropriate initial conditions. Analytically it is more convenient to make the ansatz $p = L_\mathrm{n} a_2 \sin\theta, q = L_\mathrm{n} \times a_2 \cos\theta, \xi = L_\mathrm{n}^{-1}x$. These new dimensionless variables satisfy the following hamiltonian system of equations:

$$\mathrm{d}q/\mathrm{d}\xi = \partial H/\partial p, \quad \mathrm{d}p/\mathrm{d}\xi = -\partial H/\partial q,$$

$$H(p,q,\xi) = q(p^2 + q^2 - 1) - \tfrac{1}{2}\epsilon\rho(p^2 + q^2)\sin\rho\xi, \tag{2}$$

where $\epsilon \equiv L_\mathrm{p}^{-1}L_\mathrm{i}$ and $\rho \equiv L_\mathrm{i}^{-1}L_\mathrm{n}$.

In a homogeneous medium the hamiltonian is ξ-independent, so eqs. (2) are readily solved in terms of elliptic functions [1]. If the inhomogeneity is small one can apply perturbation methods. According to the KAM theory (see, for example refs. [3,4]) the behavior of the solutions for most of the initial conditions must remain regular. However, the unperturbed system possesses a singular solution $p_0 = \tanh\xi, q_0 = 0$, in the vicinity of which the KAM theory does not hold. Mathematically this solution corresponds to a heteroclinic orbit connecting two unstable saddles $(p_\pm = \pm 1, q_\pm = 0)$. Fortunately (for those who prefer stochastic solutions to regular ones) this heteroclinic solution is of principal physical relevance: it describes the case when only one of the two waves is externally excited at $x = 0$.

The effect of a periodic perturbation on a heteroclinic solution is well known: splitting of the separatrices of the saddles $(p_\pm, q_\pm)$ gives rise to a "stochastic layer" – a region in phase space where the behavior of the solutions is extremely irregular and practically indistinguishable from a random process. Physically this means that one cannot obtain pure second harmonic generation (or its decay), provided there is only one wave at the boundary of the inhomogeneous medium: the inhomogeneity dephases the waves so that generation is succeeded by decay and then again by generation at irregular intervals along the direction of propagation.

A relative degree of stochasticity (i.e. characteristic spread of wave amplitudes and phases as well as of lengths of generation and decay processes) can be estimated by the magnitude of the separatrix splitting Δ. In the case of a small inhomogeneity ($\epsilon\rho \ll 1$) this magnitude may be analytically calculated from the formula [5,6]

$$\Delta = \epsilon\rho \int_{-\infty}^{\infty} [H_0, H_1]\, \mathrm{d}\xi, \qquad (3)$$

where $H_0(p,q) = q(p^2 + q^2 - 1)$ is the unperturbed hamiltonian, $H_1(p,q,\xi) = -\frac{1}{2}(p^2 + q^2) \sin \rho\xi$; [,] is the Poisson bracket; the unperturbed solution (p_0, q_0) must be substituted into formula (3). After the evaluation of the integral one gets

$$\Delta = \tfrac{1}{2}\pi\epsilon\rho^3 [\sinh(\tfrac{1}{2}\pi\rho)]^{-1}. \qquad (4)$$

It is remarkable that just the same expression for Δ is obtained for $\epsilon \ll 1$, ρ being arbitrary. In this case one has to perform a canonical transformation with the generating function

$$S(q,P,\xi) = [2qP - (q^2 + P^2) \sin(\epsilon \cos \rho\xi)] \times [2 \cos(\epsilon \cos \rho\xi)]^{-1},$$

the new hamiltonian has the form

$$\mathcal{H}(P,Q,\xi) = (P^2 + Q^2 - 1) \times [Q \cos(\epsilon \cos \rho\xi) + P \sin(\epsilon \cos \rho\xi)]. \qquad (5)$$

If $\epsilon \ll 1$, the hamiltonian (5) can be successfully treated by the same perturbation method, the expression obtained for Δ being exactly the same as in (4).

Thus, for the cases of $\epsilon\rho \ll 1$ and $\epsilon \ll 1$ it has been shown analytically that in the system of eqs. (2) a "stochastic layer" appears. In the case $\epsilon \approx \rho \approx 1$ there are no analytical methods but we have numerical evidence that stochasticity for these values of the parameters is retained.

Let us discuss briefly the physical meaning of the main analytical result, formula (4). For a given inhomogeneity ϵ is fixed and ρ depends on the initial externally excited intensities of the waves. For small initial amplitudes $L_n \gg L_i$ and $\rho \gg 1$. Applying the method of averaging [7] we arrive at the conclusion that the stochasticity is exponentially small. Note, that the averaged hamiltonian is $\mathcal{H}(P,Q) = J_0(\epsilon)Q(P^2 + Q^2 - 1)$, where J_0 is the Bessel function, so the dynamics of the waves is essentially governed by ϵ. With the increase of the wave amplitudes the length of the interaction decreases and at $L_n \approx L_i$ ($\rho \approx 1$) a "resonance" occurs – stochasticity is strong.

We have considered the simplest example of a nonlinear wave process – two-wave interaction. The results concerning similar effects in the case of three-wave coupling will be published elsewhere. The effects of periodical inhomogeneities on the behavior of more complex integrable nonlinear systems (for example, soliton-bearing) are also of considerable interest.

References

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