ON THE EXISTENCE OF STATIONARY MULTISOLITONS

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It is shown that nonlinear wave equations possess an infinite (but countable) set of localized stationary solutions which may be considered as stationary bound states of "elementary" solitons. The corresponding criteria are suggested and supported experimentally and numerically.

The properties of solitons as field particles are discussed intensively at present. The existence of excited bound pairs of solitons has recently been demonstrated for the sine-Gordon equation [1], the scalar field $\lambda\varphi^4$ model [2] and the generalized KdV equation [3].

Two important questions arise in this connection: (1) whether bound states with more than two solitons exist and (2) if there are stationary multisolitons, i.e. such solutions of a nonlinear wave equation which may be treated as nonexcited bound states of several "elementary" solitons. It will be shown that both questions are answered affirmatively, moreover, the number of "elementary" solitons in a bound state may be arbitrarily large.

According to ref. [3] the interaction of solitons with close velocities is analogous to the dynamics of classical particles with potential $U(s)$ where s is the distance between the solitons. The form of U is determined by an asymptotic behaviour of a single soliton field far away from its centre. Hence, the existence and number of multisolitons are defined by the number of extrema (for stable multisolitons minima) of the function $U(s)$. In the most usual cases when the asymptotic behaviour of a soliton field may be found from the linearized field equation, the general form of $U(s)$ is the following:

$$U(s) = \sum_i C_i \,\mathrm{Re}\,[\exp(\lambda_i s)]\,, \tag{1}$$

where the C_i are real constants and the λ_i are the eigenvalues of the linearized stationary equations (see eq. (3) below) with negative real parts. Two cases may be distinguished here:

(1) The eigenvalue with min $|\mathrm{Re}\,\lambda_i|$ has nonzero imaginary part. Then for large s, the potential $U(s)$ has the form of damped oscillations with an infinite number of extrema, so that an infinite number of multisolitons exist.

(2) The eigenvalue with min $|\mathrm{Re}\,\lambda_i|$ is real. Then there may be a finite number of extrema in $U(s)$, if any. So there may be either a finite number of multisolitons or none at all.

Let us illustrate this by the example of solitons described by the generalized KdV equation:

$$\frac{\partial\varphi}{\partial t} + 2\varphi\frac{\partial\varphi}{\partial x} + \sum_{i=1}^{n} \alpha_i \frac{\partial^{2i+1}}{\partial x^{2i+1}}\varphi = 0\,. \tag{2}$$

For stationary waves of the form $\varphi(x,t) = \psi(x - vt) \equiv \psi(\xi)$ we obtain after integration

$$-v\psi + \psi^2 + \sum_{i=1}^{n} \alpha_i \psi^{(2i)} = 0\,. \tag{3}$$

In the case $n = 1$ (KdV equation) both eigenvalues are real and the potential $U(s)$ does not possess any extrema, so stationary multisolitons are impossible.

We would like to discuss the case $n = 2$ in more detail [‡1]. The eigenvalues are

[‡1] The form of an elementary soliton for this case was firstly calculated by Kawahara [4].

$$\lambda_{1-4} = \pm[\pm(\alpha_1^2 + 4\alpha_2 v)^{1/2}(2\alpha_2)^{-1} - \alpha_1(2\alpha_2)^{-1}]^{1/2} . \quad (4)$$

When $\alpha_1^2 + 4\alpha_2 v < 0$ there are two pairs of complex conjugate values $\pm\text{Re}\,\lambda \pm \text{i}\,\text{Im}\,\lambda$, so $U(s) \sim \exp(-|\text{Re}\,\lambda| s)\cos((\text{Im}\,\lambda)s)$ and the number of multisolitons is infinite.

This result may be supported by more accurate evidence. In the four-dimensional phase space of eq. (3) solitons are represented by homoclinic orbits, i.e. orbits which tend to the origin for $\xi \to \pm\infty$. All orbits leaving the origin form a two-dimensional unstable manifold W^u, which corresponds to the eigenvalues $|\text{Re}\,\lambda| \pm \text{i}\,\text{Im}\,\lambda$, while the orbits tending to the origin lie on a stable manifold W^s symmetric with respect to W^u and corresponding to the eigenvalues $-|\text{Re}\,\lambda| \pm \text{i}\,\text{Im}\,\lambda$. We obtained numerically that W^u and W^s intersect transversely, their interaction line corresponding to a soliton. As was first noted by Poincaré [5] and then proved in ref. [6] the existence of a single transverse homoclinic orbit leads immediately to a countable number of intersections of W^u and W^s. A countable number of stationary multisolitons does correspond just to these intersection lines.

It should be emphasized that we have made the nonintegrability of system (2) evident. Really, if (2) were an integrable system, then eq. (3) would be integrable too [7] but this is inconsistent with the properties of homoclinic structure.

The multisolitons described by eq. (3) with $n = 2$ were observed experimentally by the authors in a chain of coupled nonlinear oscillators described earlier [3]. Eq. (2) is a continuous analog of this chain equation. Some of the observed solitons are displayed in fig. 1a.

In the case $n = 3$ a new situation is possible when the potential has a finite number of extrema. In the experiments with the corresponding chain of oscillators we did not succeed in coupling more than two elementary solitons (fig. 1b). This may be explained easily: though the "tail" of a single soliton has an extremum, that of a bound pair is already monotonous, so a third soliton cannot join it.

Eq. (2) chosen to illustrate the possibilities is also of considerable physical importance. In addition to chains of nonlinear electrical oscillators this equation for $n = 2$ describes the case of magneto-sound propagation in plasma and gravity-capillary shallow water waves (see ref. [4]). It should be noted that multisolitons may serve as the basic elements of stochastic ensembles of solitons observed in a system described by eq. (2) with $n = 2$ [8].

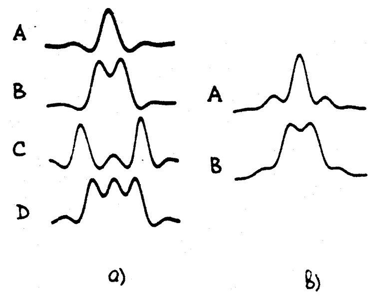

Fig. 1. "Elementary" solitons (a) and stable stationary multisolitons (b) – (d) in chains of electromagnetic oscillators, described approximately by eq. (2): (a) $n = 2$, $\alpha_1 = -3 \times 10^{-2}$, $\alpha_2 = -4.7 \times 10^{-3}$ (see refs. [3,8]); (b) $n = 3$, $\alpha_1 = 0.11$, $\alpha_2 = 0.058$, $\alpha_3 = 0.012$.

To interpret solitons as field particles the following relativistic invariant system may be used instead of eq. (3):

$$\Box\varphi_1 = \varphi_2, \quad \Box\varphi_2 = \varphi_1 - \varphi_1^2 . \quad (5)$$

Eq. (5) has the same stationary solutions as eq. (3) at $n = 2$ and, consequently, the same countable set of multisolitons. Note that in refs. [9,10] a rather wide spectrum of field particles was obtained by quantisation of a classical excited bound pair of solitons in the sine-Gordon equation. From our results the existence of a considerable variety of particles even within the frame of classical stationary solutions is evident.

Our last note is referred to three- and two-dimensional solitons, the interaction of which may apparently be described by the same approach. In this case attracting solitons would form a "planetary" pair, rotating around a common centre, while solitons with oscillating tails would rotate only in discrete orbits corresponding to the stable bound states described above.

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