Onset of stochasticity in decay confinement of parametric instability

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A dynamic system of the three resonantly coupled waves, two of which are parametrically excited, is investigated. It is shown that when the pump amplitude is increased in this system, a strange attractor (analogous to the known Lorentz attractor) is produced and corresponds to stochastic self-oscillations of the wave amplitudes, yet the phases are always fully correlated. A concrete realization of this process in a magnetoactive nonisothermal plasma is considered.

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1. INTRODUCTION

It is known that one of the mechanisms widely used to limit the intensity of parametrically amplified waves is the transfer of energy from them to waves that are damped in the linear approximation and are produced as a result of decay instability. In the simplest formulation, an investigation of this mechanism leads to the problem of the interaction of three resonantly coupled waves; this differs from the classical problem in that besides the linear damping, two of these waves possess a parametric growth rate. The starting point in this case is the following system of equations for the complex wave amplitudes:

$$\dot{a}_1=-a_2a_3-\nu_1a_1+ha_2^*,$$
$$\dot{a}_2=a_1a_3^*-\nu_2a_2+ha_1^*,\quad \dot{a}_3=a_1a_2^*-a_3, \tag{1}$$

where h is proportional to the pump amplitude.

We show in this paper, on the basis of an investigation of the solution of the dynamic system, that a stochastic stabilization regime is possible besides the trivial regime of static stabilization of the parametric instability. In such a regime the amplitudes of the interacting waves, while limited, pulsate randomly with time. In the phase space of the system (1), these stochastic motions correspond to trajectories that correspond to the so-called strange attractor.[1] The strange attractor (SA) is an attracting region in phase space, in which the behavior of the trajectories is extremely complicated and is characterized as a rule by mixing and by a continuous spectrum.

The mathematical discovery of SA and their observation in a number of dynamic systems describing real hydrodynamic experiments have led to new concepts concerning the nature of turbulence. According to these concepts, the onset in a nonequilibrium dissipative medium of disordered motions characterized by a continuous spectrum need not necessarily be due to excitation of an unusually large number of degrees of freedom of the medium. Until recently, the SA, as well as the concept of the few-mode turbulence, was associated only with systems of the hydrodynamic type. However, the results of recent papers devoted to the analysis of wave-interaction process in nonconservative media allow us to state that SA have a direct bearing also on the nature of wave turbulence in various media, particularly in a plasma.[2-5]

We derive below, with waves in a plasma as an example, equations of type (1), and then investigate analytically and qualitatively the solutions of this system. We show that the SA observed in the system (1) is similar to the Lorenz attractor known from hydrodynamics.[6,7]

2. DERIVATION OF THE BASIC EQUATIONS

By way of a concrete example we consider the parametric interaction, in a nonisothermal magnetoactive plasma, of a whistler with ion sound and with plasma oscillations near the lower hybrid resonance. An interaction of this type has attracted much attention recently in connection with the problem of RF heating of a plasma[8,9] and parametric phenomena in the ionosphere.[10] Assume that a whistler with wave vector **q** and frequency

$$\omega_{\mathbf{q}}=\omega_H c^2\omega_p^{-2}q^2\cos(\mathbf{q}\hat{\ }\mathbf{z})$$

propagates along a magnetic field **H** in a plasma. This wave can excite parametrically a plasma wave **k** with frequency

$$\omega_{\mathbf{k}}=\omega_p\omega_H(\omega_p^2+\omega_H^2)^{-1/2}\cos(\widehat{\mathbf{k}\,\mathbf{z}})$$

and ion sound $\varkappa$ with frequency $\Omega_\varkappa=c_s\varkappa$.[10] Here $\omega_p=(4\pi e^2n_0m^{-1})^{1/2}$, $\mathbf{z}\parallel\mathbf{H}$, $\omega_H=eH(mc)^{-1}$, and the ion sound is assumed to be unmagnetized. The synchronism conditions are standard:

$$\mathbf{q}=\mathbf{k}+\boldsymbol{\varkappa},\quad \omega_{\mathbf{q}}=\omega_{\mathbf{k}}+\Omega_\varkappa. \tag{2}$$

Owing to the large group velocity of its propagation, the whistler velocity can be regarded as constant. Thus, a plasma wave at the frequency of the lower hybrid resonance as well as ion sound are parametrically excited in the given pump field. The confinement of the instability is due to energy transfer to waves that are not at resonance with the pump. It is necessary to take into account first a plasma wave which is synchronous to the produced pair:

$$\mathbf{k}_1=\mathbf{k}-\boldsymbol{\varkappa},\quad \omega_{\mathbf{k}_1}=\omega_{\mathbf{k}}-\Omega_\varkappa. \tag{3}$$

It is precisely this triplet of resonantly coupled waves which will be of interest to us hereafter. Of course, under real conditions this triad will not remain isolated; moreover, since it is necessary to satisfy simultaneously the conditions (2) and (3), we shall not find a suitable wave $(\mathbf{k}_1, \omega_{\mathbf{k}_1})$ for any arbitrary growing pair. The process considered here, however, is the most elementary one and is therefore certainly worthy of attention.

Assuming spatial homogeneity, the equations for the amplitudes of interacting waves that vary slowly in time are obtained in standard fashion from the hydrodynamic equations for the RF oscillations of an electron gas and from the kinetic equations with allowance for the RF potential (Miller force) for ion sound.[11,12] The resultant equations take the simplest form if we change from the natural variables—the Fourier components of the variations of the electron densities $n_{\mathbf{k}}$ and $n_\varkappa$—to the normal amplitudes[13,14]

$$a_{\mathbf{k}}=\frac{\omega_p(2mn_0)^{1/2}}{k\omega_{\mathbf{k}}^{1/2}}\left|\frac{\omega_p^2+\omega_H^2-2\omega_{\mathbf{k}}^2}{\omega_{\mathbf{k}}^2-\omega_H^2}\right|\frac{n_{\mathbf{k}}}{n_0},\qquad b_\varkappa=\left(\frac{2n_0T_e}{\Omega_\varkappa}\right)^{1/2}\frac{n_\varkappa}{n_0}. \tag{4}$$

We then obtain

$$\begin{aligned}\dot a_{\mathbf{k}}+\gamma_{\mathbf{k}}a_{\mathbf{k}}&=-iV_{\mathbf{k}\mathbf{k}_1\varkappa}a_{\mathbf{k}_1}b_\varkappa-iW_{\mathbf{k}\varkappa\mathbf{q}}E_{\mathbf{q}}b_\varkappa^*,\\ \dot b_\varkappa+\gamma_\varkappa b_\varkappa&=-iV^*_{\mathbf{k}\mathbf{k}_1\varkappa}a_{\mathbf{k}}a_{\mathbf{k}_1}^*-iW_{\mathbf{k}\varkappa\mathbf{q}}E_{\mathbf{q}}a_{\mathbf{k}}^*,\\ \dot a_{\mathbf{k}_1}+\gamma_{\mathbf{k}_1}a_{\mathbf{k}_1}&=-iV^*_{\mathbf{k}\mathbf{k}_1\varkappa}a_{\mathbf{k}}b_\varkappa^*.\end{aligned} \tag{5}$$

Here $E_{\mathbf{q}}$ is the complex amplitude of the electric field of the whistler, $\gamma_{\mathbf{k}}$ and $\gamma_\varkappa$ are the damping decrements, for which complete equations are given in the book of Ginzburg and Rukhadze[15];

$$V_{\mathbf{k}\mathbf{k}_1\varkappa}=-\left(\frac{\Omega_\varkappa}{2n_0T_e}\right)^{1/2}\frac{\omega_p^2}{2kk_1(\omega_p^2+\omega_H^2-2\omega_{\mathbf{k}}^2)}\times\{\omega_{\mathbf{k}}(\mathbf{k}\mathbf{k}_1)-\omega_H^2\omega_{\mathbf{k}}^{-1}k_zk_{1z}+i\omega_H(z^0[\mathbf{k}\mathbf{k}_1])\},$$

$$W_{\mathbf{k}\varkappa\mathbf{q}}=-\left(\frac{\Omega_\varkappa}{\omega_{\mathbf{k}}mT_e}\right)^{1/2}\frac{e\omega_p(k_y-ik_x)}{2k}\left|\frac{\omega_{\mathbf{k}}^2-\omega_H^2}{\omega_p^2+\omega_H^2-2\omega_{\mathbf{k}}^2}\right|$$

(it was assumed in these formulas that $\omega_{\mathbf{k}}\approx\omega_{\mathbf{k}_1}\approx\omega_{\mathbf{q}}$).

The substitutions

$$\begin{aligned}a_{\mathbf{k}}&=\gamma_{\mathbf{k}_1}(|V_{\mathbf{k}\mathbf{k}_1\varkappa}|)^{-1}(1-i)\cdot 2^{-1/2}\exp[0.5i(\arg W_{\mathbf{k}\varkappa\mathbf{q}}+\arg E_{\mathbf{q}})]a_1,\\ b_\varkappa&=\gamma_{\mathbf{k}_1}(|V_{\mathbf{k}\mathbf{k}_1\varkappa}|)^{-1}(1-i)\cdot 2^{-1/2}\exp[0.5i(\arg W_{\mathbf{k}\varkappa\mathbf{q}}+\arg E_{\mathbf{q}})]a_2,\\ a_{\mathbf{k}_1}&=-\gamma_{\mathbf{k}_1}(|V_{\mathbf{k}\mathbf{k}_1\varkappa}|)^{-1}i\exp(-i\arg V_{\mathbf{k}\mathbf{k}_1\varkappa})a_3,\quad t=\tau\gamma_{\mathbf{k}_1}^{-1}\end{aligned}$$

change the system (5) to the dimensionless form (1), where $h=|W_{\mathbf{k}\varkappa\mathbf{q}}E_{\mathbf{q}}|\gamma_{\mathbf{k}_1}^{-1}$ is proportional to the pump, and $v_1=\gamma_{\mathbf{k}}\gamma_{\mathbf{k}_1}^{-1}$ and $v_2=\gamma_\varkappa\gamma_{\mathbf{k}_1}^{-1}$ are the normalized decrements. Our task is in fact the investigation of the system (1).

3. ANALYTIC INVESTIGATION OF THE SYSTEM (1). PHASE CORRELATION

A remarkable feature of the sixth-order system (1) is that it reduces to a third-order system. We demonstrate this by putting

$$\psi=\arg a_1+\arg a_2,\quad \xi=\operatorname{Im}(ha_1^*a_2^*)=-h|a_1||a_2|\sin\psi,\quad \eta=\operatorname{Im}(a_1a_2^*a_3^*).$$

It follows from (1) that

$$\frac{d}{dt}(\xi+\eta)=-(v_1+v_2)\xi-(1+v_1+v_2)\eta,$$

i.e., the projections of the system trajectories on the (ξ,η) plane enter into the angle Y between the straight lines $\xi+\eta=0$ and $(v_1+v_2)\xi+(1+v_1+v_2)\eta=0$.

It is likewise easy to deduce from (1) that

$$\dot\psi=(|a_1|^{-2}+|a_2|^{-2})(\xi+\eta).$$

In the region of Y, however, we have $\xi>0$ if $\xi+\eta>0$ and $\xi<0$ if $\xi+\eta<0$; therefore $\dot\psi>0$ if $\sin\psi<0$ and $\dot\psi<0$ if $\sin\psi>0$. It follows therefore that $\psi\to 0$, and consequently also $\xi\to 0$ and $\eta\to 0$.

This means complete correlation of the phases: they have taken on values such that the processes of parametric excitation and decay interaction have maximum intensity. We call attention to the fact that, in contrast to parameteric excitation of waves in media with a non-decay spectrum, where phase correlation takes place only for the pair due to pumping,[16] in the present case the phase of the wave a_3, which is not connected directly with the pump, also correlates.

The system (1) is invariant to the substitutions

$$a_1\to a_1e^{i\theta},\quad a_2\to a_2e^{-i\theta},\quad a_3\to a_3e^{2i\theta},$$

corresponding to the remaining leeway in the choice of the phase. We can therefore assume all the a_i to be real. Then, putting $x=a_1$, $y=a_2$, and $z=a_3$, we get

$$\dot x=hy-v_1x-yz,\quad \dot y=hx-v_2y+xz,\quad \dot z=-z+xy. \tag{6}$$

This third-order system is very similar to the popular Lorenz system that appears in the investigation of thermal convection[6] and laser dynamics[4,5]:

$$\dot X=rY-X-YZ,\quad \dot Y=\sigma X-\sigma Y,\quad \dot Z=-bZ+XY. \tag{7}$$

The system (7) is apparently the first attonomous system of ordinary differential equations, in which the SA was observed. It has attracted much attention on the part of the mathematicians and has by now been investigated in considerable detail.[17–22] We investigate below the system (6) by analogous methods.

All that can be established analytically are certain general properties of the system, as well as the local structure near the equilibrium states (ES).

The following three important properties of the system (6) are analogous to the properties of the system (7):

1. The system (6) is invariant to the substitutions

$$x\to -x,\quad y\to -y,\quad z\to z,$$

2. The phase volume shrinks uniformly:

$$\frac{\partial \dot x}{\partial x}+\frac{\partial \dot y}{\partial y}+\frac{\partial \dot z}{\partial z}=-(1+\nu_1+\nu_2),$$

so that the attractor should have a zero Lebesgue measure.

3. The confinement of the parametric instability within the framework of (6) takes place arbitrary supercriticality. In fact, let us put

$$u=2x^2+y^2+(z-3h)^2,$$

then

$$\dot u\leqslant -ku+9h^2,\quad k=\min(2\nu_1, 2\nu_2, 1),$$

i.e., all the trajectories are contained in the ellipsoid $u\leqslant 9h^2k^{-1}$.

We consider now the ES, and pay attention to their evolution with increasing pump h. At $h<(\nu_1\nu_2)^{1/2}$ there is a single equilibrium state $O(0,0,0)$ to which all the trajectories are attracted. At $h>(\nu_1\nu_2)^{1/2}$ the threshold of parametric excitation is exceeded and two nonzero ES appear: $C^{\pm}(\pm(z^0l)^{1/2},\pm(z^0l)^{1/2},z^0)$, where $z^0=(h^2-\nu_1\nu_2)^{1/2}$, $l=(h-z^0)\nu_1^{-1}$, and correspond to the static regime of instability elimination. The zero-point ES then becomes unstable.

We call attention to the high effectiveness of the confinement of the instability within the considered triplet. Thus, the dispersion relations admit of the decay of the plasmon $(\mathbf{k},\omega_{\mathbf{k}})$ into a pair of waves not connected with the pump[13]:

$$\mathbf{k}=\mathbf{k}_2+\varkappa_1,\quad \omega_{\mathbf{k}}=\omega_{\mathbf{k}_2}+\Omega_{\varkappa_1},$$

but for this purpose it is necessary to exceed the plasmon threshold amplitude, which can be estimated in dimensionless variables at $\nu_2^{1/2}$. But at any h we have $lz^0<\nu_2$, so that the ES $C^{\pm}$ are stable to excitation of an "extraneous" pair. We note that this conclusion can be arrived at only for the static confinement regime; in the stochastic regime the question remains open.

At $\nu_2<\nu_1+1$ the ES $C^{\pm}$ are always stable. This is precisely the case in decay (with participation of ion sound) confinement of Langmuir waves excited by an electromagnetic wave in an isotropic plasma.[23] In this stiuation x and y are plasmons parametrically connected with the pump and having close frequencies, therefore $\nu_2\approx\nu_1$.

In the case considered by us, that of wave interaction in a magnetoactive plasma, the plasmons x and z are practically equally damped: $\nu_1\approx 1$, and the condition $\nu_2<\nu_1+1\approx 2$ may be violated at sufficiently large damping y of the ion sound. Then the ES $C^{\pm}$ becomes unstable at

$$h>h^0=\left[\frac{f+(f^2+64g\nu_1\nu_2)^{1/2}}{2g}\right]^{1/2},$$

$$g=4[(\nu_1-\nu_2)^2-1],$$

$$f=(\nu_2-\nu_1)^2[(1+\nu_1+\nu_2)^2+8\nu_1\nu_2]+8\nu_1\nu_2(\nu_1+\nu_2).$$

We note that in a collisionless plasma the plasma-wave damping decrements can be quite small, and then $\nu_2\gg\nu_1$. In this case $h^0=0.5\nu_2$, and in terms of the initial variables $|E_{\mathbf{q}}^0|>\gamma\,(2|W_{\mathbf{k}\varkappa\mathbf{q}}|)^{-1}$, i.e., the pump wave amplitude at which the ES $C^{\pm}$ become unstable does not depend on the plasmon damping decrement.

Thus, at sufficiently large supercriticalities, all the ES of the system (6) are unstable. They have the following structures:

ES O—saddle-node with two-dimensional stable separatrix (corresponding to two negative eigenvalues) and two one-dimensional unstable ones (positive eigenvalue).

ES $C^{\pm}$—saddle-foci. The trajectories approach them along one-dimensional separatrices and move away unwinding along two-dimensional separatrices (corresponding to complex-conjugate eigenvalues with positive real parts).

4. NUMERICAL INVESTIGATION OF THE SYSTEM (6)

A numerical investigation of the system (6) has shown complicated and entangled trajectories exist in it at $h>h^0$. A typical realization appears outwardly as follows: the generating point in phase space makes several revolutions around the ES C^+, then goes over to C^- and rotates around it, returns back to C^+, etc. (see Fig. 1).

It is convenient to carry out the investigation with the dimensionality of the phase space decreased. This is done, first, by constructing a two-dimensional mapping of the sequence and, second, by using the construction of the inverse limit to reduce it to a one-dimensional mapping (this was done by others[17-22] for the Lorenz system).

For the secant plane we take the two-dimensional set Σ—that part of the plane $z=z^0$ on which $\dot z<0$. The mapping of the sequence Ψ: $\Sigma\to\Sigma$, which sets the initial point $\sigma_i\in\Sigma$ in correspondence with the point $\sigma_{i+1}\in\Sigma$, at which the trajectory that begins at σ_i returns to Σ for the first time, was constructed at $\nu_1=1$, $\nu_2=4$, and h

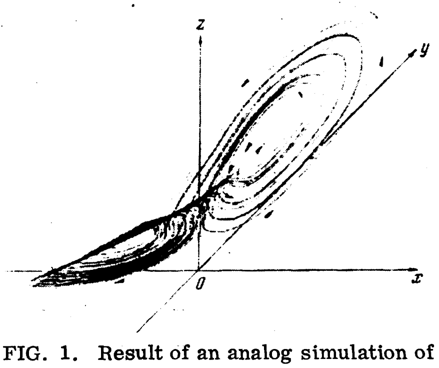

FIG. 1. Result of an analog simulation of the system (6) at $\nu_1=1$, $\nu_2=4$, and $h=6.75$.

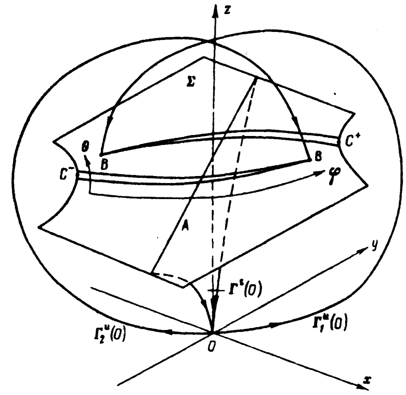

FIG. 2. Mapping of the sequence for the system (6).

=5.875 (Fig. 2).

The principle role in the organization of the stochasticity is played by the ES O, whose two-dimensional stable separatrix $\Gamma^s(O)$ crosses Σ along the line A. The mapping of Ψ is discontinuous: points lying on opposite sides of A go off along different separatrices $\Gamma^u_{1,2}(O)$. The operation Ψ is a combination of stretching in the φ direction with compression in the θ direction (the compression is undervalued in the figure for the sake of clarity).

If we consider the action of Ψ along φ, then we obtain a one-dimensional mapping for which Ψ is the inverse limit.[21,22] It will be convenient, following Lanford,[17] to identify the points that are symmetrical with respect to rotation about the z axis; we then obtain a continuous mapping of the segment into itself $\varphi \to \Phi(\varphi)$. It is shown on Fig. 3d. The points a, b, and c correspond to the line A, to the points $B=\Psi(A)$, and to $C^{\pm}$.

It is well known[24] that similar stretching mappings[1)] have a stochastic behavior, in particular, mixing properties. The mixing for the Ψ mapping was proved by Bunimovich and Sinai.[13] The SA structure in the system (6) is analogous to the SA of the Lorenz system (7). A numerical analysis has shown that the SA of the system (6) belongs to the single-parameter family of attractors described by Guckenheimer and Williams.[20-22] The structure of these SA is topologically quite complicated and is sensitive to the parameters of the system (non-coarseness).

We now trace the formation of the SA with increasing h, at fixed $\nu_1=1$ and $\nu_2=4$. At $2<h<4.0$, Φ takes the form shown in Fig. 3a. The separatrix $\Gamma^u_1(O)$ goes to C^+: $b<a$. At the point $h\approx 4.0$ bifurcation takes place: $\Gamma^u_{1,2}(O)\subset\Gamma^s(O)$, and here $b=a$. The form of Φ at $4.0 \lesssim h \lesssim 4.84$ is shown in Fig. 3b. At the instant of bifurcation there appears a cycle (point d) and simultaneously the periodic points of the period 3, and this leads to the stochasticity.[25] But this stochasticity (which constitutes the vicinity of a homoclinic contour on two symmetrical cycles) is not attracting, since almost all the points from the region $d<\varphi<b$ are attracted to c.

At $h\approx 4.84$ we have $d=\Phi(b)$ and the points cease to go over to c from the region $d<\varphi<b$, that is to say, at $4.84 \lesssim h < h^0 \approx 4.92$ the system has two attractors, two simple and one strange, and the region of their attraction is separated by the cycle d (Fig. 3c). At $h=h^0$ the ES $C^{\pm}$ become unstable (d sticks to c) and only the SA remains. Thus, the onset of stochasticity takes place in hard fashion and is accompanied by hysteresis. At $h\approx 13.4$ stable limit cycles appear in the system (6). The obtained stochastic-regime criteria corresponding to the appearance of the SA can apparently be realized in experiment without difficulty. In fact the transition to the SA in the example considered above ($\nu_1=1$ and $\nu_2=4$) takes place when the instability threshold is exceeded by a factor 2.5 ($h^0\approx 5$). Such values of the pump-wave amplitude are now readily attainable both in a laboratory plasma and in the ionosphere. In particular, as shown in,[10] the conditions for parametric instability of a whistler in the ionosphere can be realized in the operation of radio stations in the ultralong-wave band. Since the properties of the ionosphere vary with height, different parametric-instability regimes will take place at the corresponding heights.

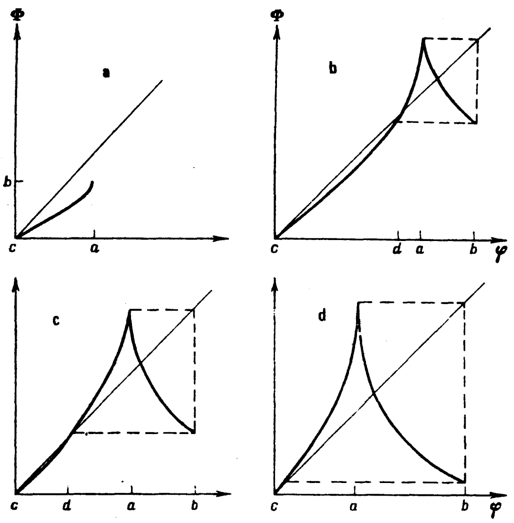

FIG. 3. One-dimensional mappings of the sequence at $\nu_1=1$ and $\nu_2=4$: a) $h=3.5$, b) $h=4.5$, c) $h=4.875$, d) $h=5.875$.

5. CONCLUSION

The model considered above for the confinement of parametric instability presupposes the existence of a wave that is in synchronism with the pair excited by the pump. This is a rather frequent situation and can be encountered, besides in the case considered here, when waves are excited in a magnetoactive plasma near the upper hybrid resonance, as well as in an isotropic non-isothermal plasma. In this case Eqs. (1) describe the mechanism that is the most effective, in our opinion, for instability confinement and leads to the onset of stochasticity even in a system of three interacting waves. We point out also that stochastization of the amplitudes occurs in the case of complete phase correlation, i.e., the situation here is to some degree the converse of the frequently employed random-phase approximation.

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1) Although the mapping of Fig. 3d has sections with $|d\Phi/d\varphi| < 1$, this mapping becomes uniformly stretching if the metric is properly chosen.

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Indirect multispin exchange

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If the indirect exchange between f-spins is effected by s-electrons with nonzero total spin, then it is of essentially non-Heisenberg character. For the particular case of indirect exchange via a donor electron in a magnetic semiconductor it is possible to construct an equivalent magnetic Hamiltonian having the form of the square root of an expression bilinear in the f-spins. The Ruderman-Kittel term is a small correction to it. The constructed Hamiltonian accounts for the spectrum of the system accurately, but the average values of the spin operators can be expressed in terms of its eigenfunctions only in a manner that is, generally speaking, different from the manner accepted in quantum mechanics. The Hamiltonian contains all the spin invariants possible for isotropic systems: multispin, biquadratic, etc. The spin-spin interaction is noncentral. With the aid of this Hamiltonian the localized magnons in a ferromagnetic semiconductor are investigated.

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1. INTRODUCTION

As is well known, the isotropic exchange interaction is accurately described by the Heisenberg Hamiltonian only in the case of a system consisting of two spin-$\frac{1}{2}$ magnetic atoms. If the spin, S, of these atoms exceeds $\frac{1}{2}$, the exchange between them is described by a Hamiltonian that is a polynomial of degree $2S$ in the scalar product, $\mathbf{S}_1 \cdot \mathbf{S}_2$, of the spins.[1] Even more complex is the situation in the case of a large number of atoms, when into the exchange Hamiltonian enter multi-spin terms of the type $\mathbf{S}_g \cdot \mathbf{S}_f \ldots \mathbf{S}_k \cdot \mathbf{S}_h$. Although for many physical systems the Heisenberg term in the magnetic Hamiltonian is the dominant term, in certain cases the non-Heisenberg terms are not small. Of the non-Heisenberg Hamiltonians only those that are linear combinations of quadratic and biquadratic terms (i.e., $\mathbf{S}_g \cdot \mathbf{S}_f$ and $(\mathbf{S}_g \cdot \mathbf{S}_f)^{[2]}$; see, for example, Refs. 2–4) have been investigated in detail. Hamiltonians with four-spin terms, $(\mathbf{S}_g \cdot \mathbf{S}_f)(\mathbf{S}_k \cdot \mathbf{S}_h)$, added to the Heisenberg terms have also been investigated.[5]

In the present paper we shall show that in certain physical systems the isotropic exchange interaction is described by a Hamiltonian of a type entirely different from the type indicated above.1) These are systems in which the indirect exchange between the localized f-spins is effected by mobile s-electrons that are completely polarized with respect to spin. Such a situation differs sharply from the indirect exchange in systems

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