# A simple autogenerator with stochastic behavior

A. S. Pikovskii and M. I. Rabinovich

*Gor'kii Radiophysics Scientific Research Institute*

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In recent years new ideas have appeared about the nature of turbulence as stochastic autooscillations possible only in dynamical systems with a small number degrees of freedom ($n \geq 1.5$) (Refs. 1-6). These ideas stimulated the search for simple models of nonlinear processes in nonconservative systems (media) which would display stochastic behavior. Some models of this type are now known and the simplest of them are described by third-order equations. It was shown fairly convincingly by means of numerical modeling and qualitative analysis that these models show stochastic behavior.[2,6] As far as we know, for any of these models the stochastic behavior has not been proved rigorously, even in the asymptotic sense. In the present work we construct another simple model which may show stochastic behavior. This model is an autogenerator similar to the Van der Pol generator but containing an additional active element.

The system in question with one and a half degrees of freedom contains a small parameter $\mu$ at the highest derivative. When $\mu = 0$ the analysis of the autogenerator dynamics reduces to the analysis of a point mapping of a segment into itself. From the properties of this mapping it follows that in the phase space of the system there is an attractive region – attractor – where there are no stable equilibrium states or stable periodic motions. For the motion in this region of the phase space there is a natural invariant probability distribution with respect to which the dynamical system is ergodic. It can therefore be verified in this way that for $\mu = 0$ this autogenerator shows stochastic behavior, i.e. It is a true noise generator.

The diagram of the autogenerator is shown in Fig. 1a. It includes a negative conductance $-g$ and a tunnel diode whose characteristic is shown in Fig. 1b. In dimensionless variables

$$x = I/I_0 - 1, \quad y = I_0^{-1}[(U_0 - U)C^{1/2}L^{-1/2} + I],$$
$$z = U/U_0 - 1, \quad \tau = t(LC)^{-1/2}$$

this autogenerator is described by the equations

$$\begin{aligned} \dot{x} &= y - \delta z, \\ \dot{y} &= -x + 2\gamma y + \alpha z + \beta, \\ \mu \dot{z} &= x - f(x), \end{aligned} \tag{1}$$

where

$$\delta = C^{1/2}L^{-1/2}U_0/I_0, \quad \gamma = 0.5(gL - Cr)(LC)^{-1/2}, \quad \beta = gU_0/I_0 - 1,$$
$$\mu = \delta C_1/C, \quad \alpha = 1 + \beta - 2\gamma\delta,$$

and f(z) is the idealized characteristic shown in Fig. 1b by the dashed line. For $\mu \ll 1$ all motion of the system (1) can be separated into fast (along the lines x = const

  

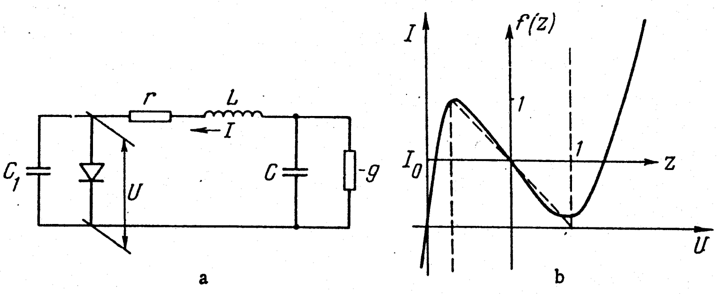


FIG. 1. a) Diagram of the generator; b) volt-ampere characteristic of the tunnel diode and its piecewise linear approximation (dashed line).

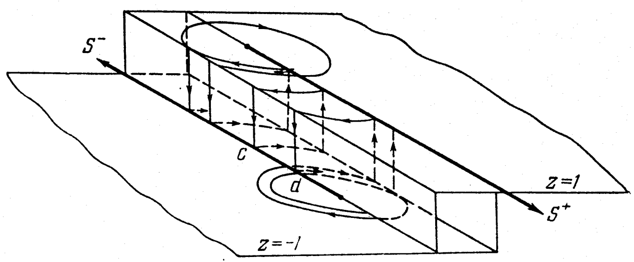


FIG. 2. The phase space of the system (1) when there are two symmetric attractors.

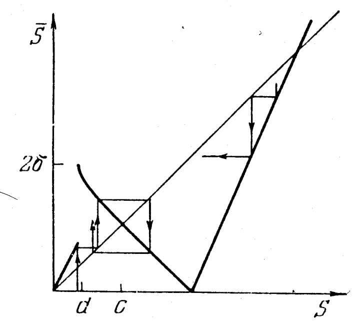


FIG. 3. The Poincare mapping generated by the trajectories of the phase space shown in Fig. 2.

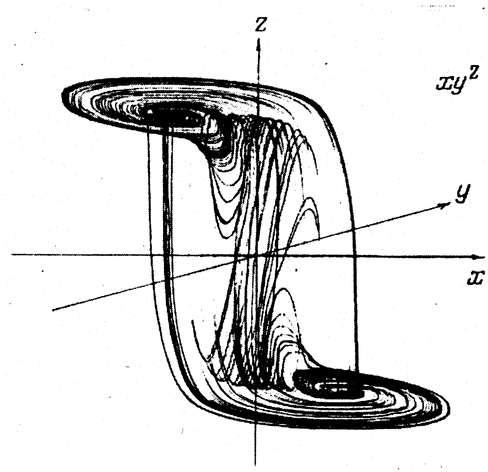


FIG. 4. The form of the symmetric singular attractor obtained by analog modeling of the system (1).

and y = const) and slow (in the planes $z = -1$, $x < 1$ and $z = 1$, $x > -1$). For $\mu \to 0$ the phase space of the system (1) therefore degenerates into two half-planes A and B overlapping in the strip $-1 < x < 1$, and the crossing of a point from one half-plane to the other takes place on the lines $x = 1$ and $x = -1$. Without any limitation in generality we can put $\beta = 0$, and the system becomes invariant with respect to the transformation $x \to -x$, $y \to -y$, $z \to -z$.

The investigation of the motion of the system (1) reduces to the analysis of the Poincare transformation of the set $S = S^+ + S^-$ into itself ($S^-$ is the half-line $x = -1$, $z = -1$, $y > -\delta$, and $S^+$ is the half-line $x = 1$, $z = 1$, $y < \delta$). This transformation is discontinuous since all trajectories beginning and ending in S can be divided into two groups: 1) Those lying in only one half-plane. For these the Poincare transformation is

$$\bar{S} = T_1 S = \exp(2\pi\varkappa) S; \tag{2}$$

2) those that cross from one half-plane to the other. For these

$$\bar{S} = T_2 S, \tag{3}$$

where $T_2$ is given parametrically by

$$\begin{gathered} S = \omega \cdot \exp(-\varkappa\tau)/\sin\tau, \\ \bar{S} = 2\delta - \omega(\cos\tau + \varkappa\sin\tau)/\sin\tau, \\ \omega = (1-\gamma^2)^{1/2}, \quad \varkappa = \gamma/\omega, \end{gathered} \tag{4}$$

and S and $\bar{S}$ in (4) belong to different half-planes (Fig. 2).

The form of the transformation (2)-(4) is clear from Fig. 3. It is qualitatively different in two cases $c > T_1 d$ and $c < T_1 d$, where $d = (T_2^{-1})2\delta$ and $c = T_2 c$ [sic]. The trajectory of each point is given by the sequence of transformations

$$\ldots (T_1)^{p-1}(T_2)^{q-1}(T_1)^{p_0}(T_2)^{q_0}(T_1)^{p_0}(T_2)^{q_1} \ldots \tag{5}$$

In the first case all the $q_i$'s are even. This means that there are two symmetric attractors in the phase space (Fig. 3). Inside the attractors the trajectories move in a spiral around one of the unstable foci and after crossing to the other half-plane they return in the opposite direction. In the other case the $q_i$'s can be odd, there is only one attractor, and its trajectories now fall in the neighborhood of both equilibrium states (Fig. 4).

It follows immediately from (2)-(4) that the derivative (where it exists) satisfies the inequality $|d\bar{S}/dS| > 1$, from which is follows that this transformation has no stable periodic points. In addition, this transformation satisfies the Kosyakin–Sandler theorem.[7] According to this thorem, the transformation has an invariant measure with respect to which the transformation will be ergodic. Consequently, the attractors can be called singular.[1,3] In the phase space of the system (1) therefore exists an attractive region where there are no stable equilibrium states or limiting cycles, the motion is ergodic, and the oscillations which set in will be stochastic.

It is clear that the development of stochastic autooscillations and their properties will depend weakly on the way in which the nonlinear characteristic of f(z) is idealized. This was verified by analog modeling of the system (1). The results of the computer experiment for $f(z) = z^3 - z$, $\beta = 0$, and $\mu = 0.1$ are given in Fig. 4. For $\delta = 0.43$, $\alpha = -0.013$, and $\gamma = 0.3$ we observed a stochastic behavior of type $a$ (two independent attractors containing the neighborhood of only one equilibrium state), and for $\delta = 0.66$ $\alpha = 0.33$, and $\gamma = 0.35$ we observed stochastic behavior of type b (only one "symmetric" attractor).

Stochastic autooscillations are possible also in second-order dynamical systems if these contain nonlinearities of the relay type. These systems also have a two-sheet phase space[8] and if there are active elements present, the behavior of these systems can be similar to the behavior of system (1).

We emphasize that the possibility of investigating the stochastic behavior in a system with a small parameter at the highest derivative by mapping a segment into itself was realized in principle by Rossler,[9] who, however, was unable to carry out this program.

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