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PHASE SYNCHRONIZATION IN REGULAR AND CHAOTIC SYSTEMS

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In this contribution we present a brief introduction to the theory of synchronization of self-sustained oscillators. Classical results for synchronization of periodic motions and effects of noise on this process are reviewed and compared with recently found phase synchronization phenomena in chaotic oscillators. The basic notions of phase and frequency locking are reconsidered within a common framework. The application of phase synchronization to data analysis is discussed.

1. Introduction

The history of synchronization goes back to the 17th century when the famous Dutch scientist Christiaan Huygens [1673] reported on his observation of synchronization of two pendulum clocks. The systematic study of this phenomenon, experimental as well as theoretical, was started by Edward Appleton [1922] and Balthasar van der Pol [1927]. They showed that the frequency of a triode generator can be entrained, or synchronized, by a weak external signal with slightly different frequency. These studies were of high practical importance because such generators became basic elements of radio communication systems.

The next impact to development of the theory of synchronization was given by representatives of the Russian school. Andronov and Vitt [1930a, 1930b] further developed methods of van der Pol and generalized his results. The case of $n : m$ external synchronization was studied by Mandelshtam and Papaleksi [1947]. Mutual synchronization of two weakly nonlinear oscillators was analytically treated by Mayer [1935] and Gaponov [1936]; relaxation oscillators were studied by Bremsen and Feinberg [1941] and Teodorchik [1943]. An important step was made by Stratonovich [1958, 1963] who developed a theory of external synchronization of a weakly nonlinear oscillator in the presence of random noise.

The development of rigorous mathematical tools of the synchronization theory started with works on circle map [Denjoy, 1932] and forced relaxation oscillators [Cartwright, 1948; Cartwright & Littlewood, 1945]. Recent development has been highly influenced by Arnold [1961] and Kuramoto [1984].

Reviews of synchronization phenomena as well as original results can be found in monographs of Teodorchik [1952], Hayashi [1964], Malakhov [1968], Blekhman [1971, 1981], Landa [1980, 1996], Romanovsky *et al.* [1984] and Kuramoto [1984].

We start this tutorial with the description of classical results for the synchronization of a periodic oscillator by external force (Sec. 2). We discuss here the notions of phase and frequency locking, and then, in Sec. 3, we show what changes if fluctuations are taken into account. Next, we demonstrate how these notions can be generalized to be useful for chaotic systems as well (Sec. 4). We end with a description of how the idea of phase synchronization can be used to detect the weak interaction between oscillators from observed data (Sec. 5). For the sake of simplicity we restrict ourselves here to the case of external forcing.

2. Entrainment of a Periodic Oscillator

Stable periodic self-sustained oscillations of an autonomous dissipative dynamical system are represented by a stable limit cycle in its phase space. If the oscillator is forced externally, this simple dynamics is generally destroyed. It is important, that if the force is small, its influence can be described in a rather universal way. The first step towards this description is to introduce new variables of the unforced system: the phase and amplitudes. The phase is a variable that corresponds to the motion along the limit cycle, i.e. along the direction where neither contraction nor expansion of the phase volume occurs. Therefore, this direction in the phase space and, respectively, *the phase of oscillations corresponds to the zero Lyapunov exponent.* A natural way to define the phase is to take it proportional to time and increase by 2π during one period of the oscillation T_0.[1] Then the dynamics of the phase on the cycle can be described as

$$\frac{d\phi}{dt} = \omega_0 \,, \tag{1}$$

where $\omega_0 = 2\pi/T_0$. Amplitudes are all other variables of the dynamical system that are locally transversal to the cycle; they correspond to the negative Lyapunov exponents.

The description in terms of Lyapunov exponents clearly demonstrates why the phase is an exceptional variable of a dynamical system: being correspondent to the sole neutrally stable direction, the phase, in contrast to amplitudes, can be controlled already by a very weak external action. Indeed, a weakly perturbed amplitude will relax to its stable value, whereas a small perturbation of the phase neither grows nor decays. Thus, even very small phase perturbations can be easily accumulated.

So far the phase is defined only on the limit cycle, but not in its vicinity. One way to extend this definition is to demand that Eq. (1) is valid for the phase not only on the cycle, but in its neighborhood as well; this phase is also denoted by ϕ. Such a definition implies that the transversal hypersurfaces of constant phase are the isochrones, i.e. they are invariant if the dynamics is observed stroboscopically with the period of oscillations T_0 [Kuramoto, 1984]. In physics, one often speaks of nonisochronicity as of the dependence of the oscillatory frequency on the amplitude. In Eq. (1) the frequency is constant, and the nonisochronicity means that the hypersurfaces of constant phase are not orthogonal to the cycle, but cross it at some angle $\neq \pi/2$. Because hypersurfaces of constant phase form a foliation of a neighborhood of the cycle, the correct phase ϕ can be obtained from any other cyclic (phase-like) variable θ via some transformation $\phi = \phi(\theta, A)$, where A denotes the amplitude variables. Sometimes, to characterize synchronization of a particular system, it is advantageous to use θ: This variable can be estimated from data and the mean observed frequencies of forced or coupled oscillators obtained by means of ϕ and θ coincide: $\Omega = \langle\dot{\phi}\rangle = \langle\dot{\theta}\rangle$. For a theoretical treatment the phase ϕ is more convenient.

The difference in the relaxation time scales of perturbations of the amplitudes and the phase[2] allows one to describe the effect of small periodic external force with a single phase equation. Indeed, making a perturbation expansion, one can see that, due to stability property, deviations of the amplitudes are small, while deviations of the phase can be large (albeit slow). As a result, one can derive (see [Kuramoto, 1984] for details) the phase equation

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon Q(\phi, \varphi) \,, \tag{2}$$

where ε is a small parameter proportional to the amplitude of the force, φ is its phase obeying $\dot{\varphi} = \omega = 2\pi/T$, and Q is a 2π-periodic both in ϕ and φ function. Equation (2) describes dynamics on a torus $0 \le \phi < 2\pi$, $0 \le \varphi < 2\pi$. Taking the Poincaré map at $\varphi = 0$, we can reduce Eq. (2) to a circle map

$$\phi_{n+1} = \phi_n + F(\phi_n) \tag{3}$$

with a 2π-periodic function F. The dynamics of the map (3) is characterized by the rotation number (see, e.g. [Ott, 1992])

$$\rho = \lim_{N\to\infty} \frac{\phi_N - \phi_0}{2\pi N} \,.$$

This number is nothing other than the ratio between the observed frequency of oscillations Ω and the frequency of the external force ω:

$$\rho = \frac{\Omega}{\omega} \,, \quad \text{where } \Omega = \langle\dot{\phi}\rangle \,.$$

[1]Sometimes we consider the phase defined on the entire real line, and sometimes as a cyclic variable that is defined on the $[0, 2\pi]$ interval. We hope that the use of a particular definition can be always understood from the context.

[2]For the amplitudes, the relaxation time of perturbation is determined by the inverse of Lyapunov exponents; perturbations of the phase do not decay at all.

If the rotation number is rational, the observed frequency is in the rational relation with the frequency of the external force

$$\Omega = \frac{m}{n}\omega$$

and this regime is called $m : n$ synchronization.

We now discuss the above definition in more detail, taking for simplicity the case of 1:1 synchronization. Usually, this phenomenon is understood as the appearance of a certain relation between phases, or as *phase locking*. In the literature, this notion is used in different senses. Most restrictively, one speaks of phase locking if the phases of two or more oscillators coincide exactly. In our case it would mean $\phi = \varphi$. More generally, a constant phase shift is allowed: $\phi = \varphi + \text{const}$. Both these definitions mean that the phase of the oscillator rotates uniformly with the frequency of the external force. From Eq. (2), however, it follows that we cannot expect these properties to be valid even for small forcing. Indeed, Eq. (2) admits the solution $d\phi/dt = \omega$ only in a particular case when the coupling function Q depends on the phase difference only: $Q(\phi, \varphi) = q(\phi - \varphi)$. To illustrate this, let us introduce the phase difference $\psi = \phi - \varphi$ and rewrite Eq. (2) as

$$\frac{d\psi}{dt} = \omega_0 - \omega + \varepsilon q(\psi)\,. \qquad (4)$$

In the synchronous state this equation should have (at least one) stable point. This happens if the frequency mismatch (detuning) is small enough, $\varepsilon q_{\min} < \omega - \omega_0 < \varepsilon q_{\max}$, and this condition determines the synchronization (phase-locking, mode-locking) region on the (ω, ε) plane. Within this region, the phase difference remains constant, $\psi = \delta$, and the value of this constant depends on the detuning, $\delta = q^{-1}[(\omega - \omega_0)/\varepsilon]$ (here the stable branch of the inverse function should be chosen).

Generally, the coupling function $Q(\phi, \varphi)$ cannot be reduced to a function of the phase difference ψ. Then, even in a synchronous regime ψ is not constant but fluctuates, although these fluctuations are bounded. Thus, we can define phase locking according to relation

$$|\phi - \varphi - \delta| < \text{const}\,, \qquad (5)$$

from which the condition of frequency locking $\Omega = \langle\dot{\phi}\rangle = \omega$ naturally follows.

The latter definition of phase locking will be used in the treatment of chaotic oscillations (Sec. 4), but even for periodic regimes it has an advantage when the forced oscillations are not close to the original limit cycle.

As an example we consider synchronization transition in a periodically forced weakly nonlinear oscillator (note, that now the forcing is not necessarily small). Additionally, we assume for simplicity that the autonomous oscillator is isochronous, so that ω_0 does not depend on the amplitude A. As we are looking for synchronous regimes, it is convenient to seek for the solution $u(t)$ at the frequency of the external force

$$u(t) = \text{Re}(Ae^{i\phi}) = \text{Re}(Ae^{i\psi}e^{i\omega t}) = \text{Re}(a(t)e^{i\omega t})\,. \qquad (6)$$

For the complex amplitude $a(t)$ one obtains averaged (truncated) equations (see, e.g. [Bogoliubov & Mitropolsky, 1961; Glendinning, 1994])

$$\dot{a} = -i\nu a + a - |a|^2 a - i\varepsilon\,. \qquad (7)$$

Here $\nu = \omega - \omega_0$ is the frequency mismatch, and ε is the (renormalized) amplitude of the force. For small ν, Eq. (7) has a solution $a(t) = Ae^{i\psi} = \text{const}$ that corresponds to the synchronous regime when the phases of the oscillator and external force are locked with the constant phase shift, $\phi - \varphi = \delta = \text{const}$. With increase of the frequency mismatch, a transition out of the synchronous state occurs. The form of this transition essentially depends on the amplitude of the external force; we discuss the essential features below (for details of bifurcation analysis see [Holmes & Rand, 1978; Argyris *et al.*, 1994]).

2.1. *Synchronization transition at small amplitudes of the external force*

Fixing the parameter ε at a small value ($\lesssim 0.6$), we start from the synchronous state ($\nu \approx 0$) and increase the absolute value of the detuning $|\nu|$ until we observe a loss of synchronization, as is illustrated in Fig. 1. First, there are three fixed points: one unstable focus, one stable node and one unstable saddle. With the increase of $|\nu|$ the saddle and node come closer and eventually collide at the bifurcation point, giving birth to the limit cycle. The rotation velocity of the phase increases smoothly, amplitude modulation is small (Fig. 2). This transition can be as well described by Eq. (2), i.e. using the phase approximation. There a stable and unstable cycle

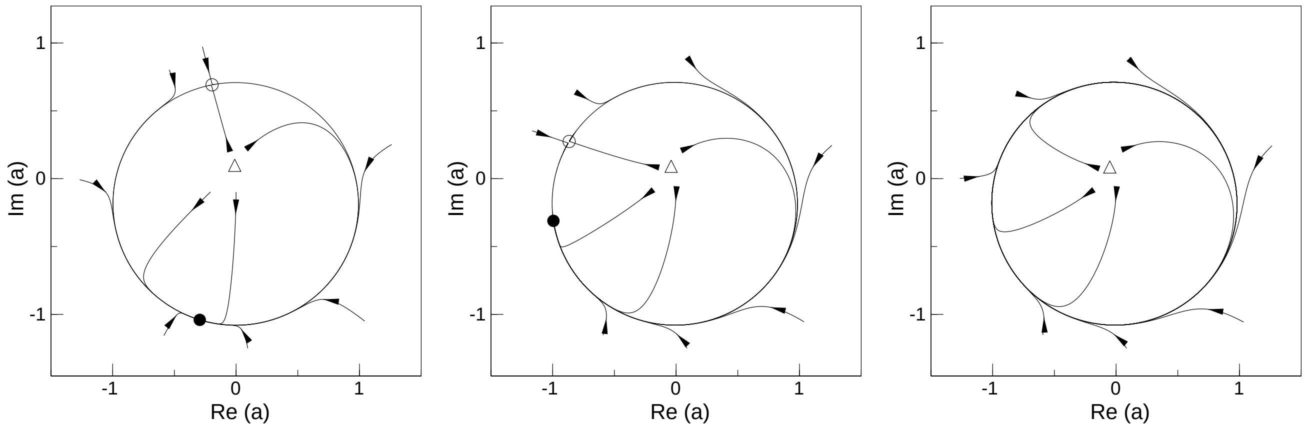

Fig. 1. Loss of synchronization via a saddle-node bifurcation. (a) In the middle of the synchronization region there exist an unstable focus (shown by triangle), a stable (filled circle) and an unstable (open circle) fixed points. (b) Stable and unstable fixed points come closer near the border of synchronization. (c) A stable limit cycle exists outside the synchronization region; it is born from an invariant curve formed by the unstable manifolds of the saddle.

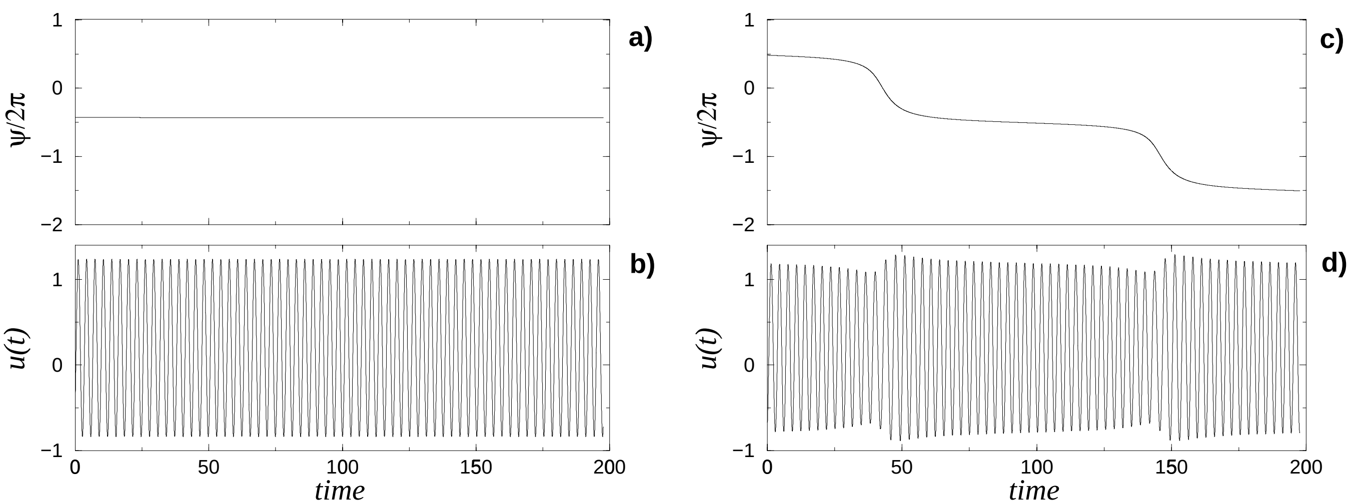

Fig. 2. Oscillations in the forced oscillator (7) at small forcing amplitudes. Inside the synchronization region the amplitude and the phase difference are constant (a, b). Just after the transition out of synchronization the phase difference ψ rotates nonuniformly, epochs of nearly constant ψ are intermingled with 2π-slips (c); the amplitude is slightly modulated.

annihilate and a quasiperiodic motion on the torus appear. This is not surprising, as Eq. (2) should be universally valid for very small forcing amplitudes.

2.2. *Synchronization transition at large amplitudes of the external force*

Now we fix the parameter ε at a large value ($\gtrsim 0.6$) and vary ν. The transition from synchronous state occurs via the Andronov–Hopf bifurcation (Fig. 3). In the middle of the synchronization region there is a stable node. When $|\nu|$ increases, it becomes a stable focus. At the transition point it loses stability, and a stable limit cycle appears. First, the amplitude of the limit cycle is small, so a point on it does not rotate around the origin. It means that the process $u(t)$ has the amplitude and the phase modulation, but the frequency of the oscillator remains the same as that of the external force [see Fig. 4(a)]. The phase difference is not constant any more, but remains bounded, so that the phase locking condition (5) is fulfilled. The situation changes if the cycle envelopes the origin [Fig. 4(b)]. Now the phase difference ψ rotates and the observed frequency $\langle\dot{\phi}\rangle \neq \omega$.

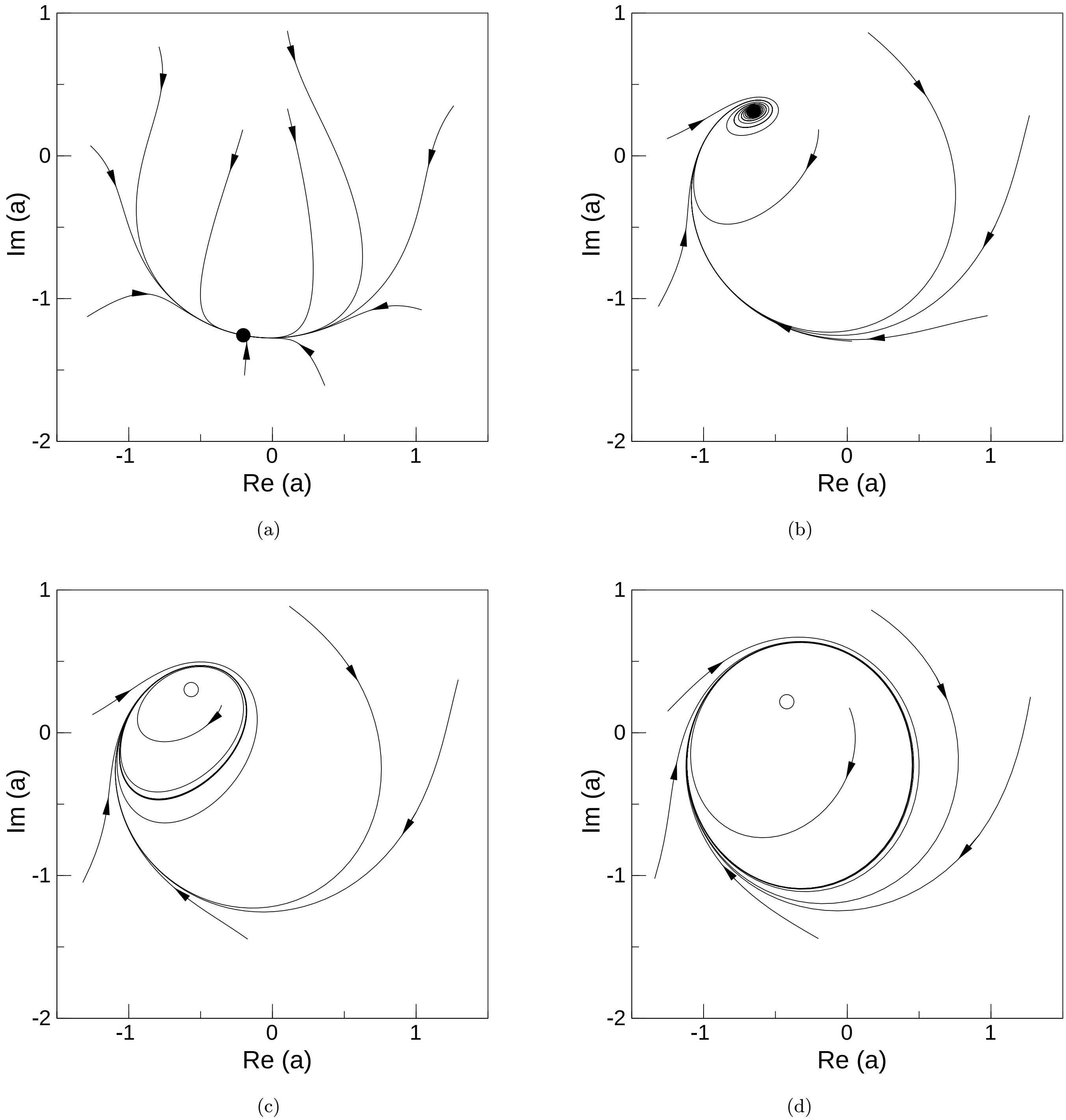

Fig. 3. Loss of synchronization via the Andronov–Hopf bifurcation. (a) Near the center of the synchronization region $\nu \approx 0$ all trajectories are attracted to a stable node. (b) Near the boundary of synchronization the fixed point is of focus type. (c) A stable limit cycle appears via Hopf bifurcation, however, this cycle does not envelope the origin, so that the observed frequency is still the same as that of the external force. (d) As the amplitude of the limit cycle grows, it envelopes the origin and the synchronization breaks.

A phase diagram of different regimes in Eq. (7) is shown in Fig. 5, where the lines of saddle-node and Andronov–Hopf bifurcations are shown. The transition from region D to region C [i.e. from Fig. 4(a) to Fig. 4(b)] is not a bifurcation, it can be characterized as the state where the amplitude of forced oscillations vanishes at some moment of time.

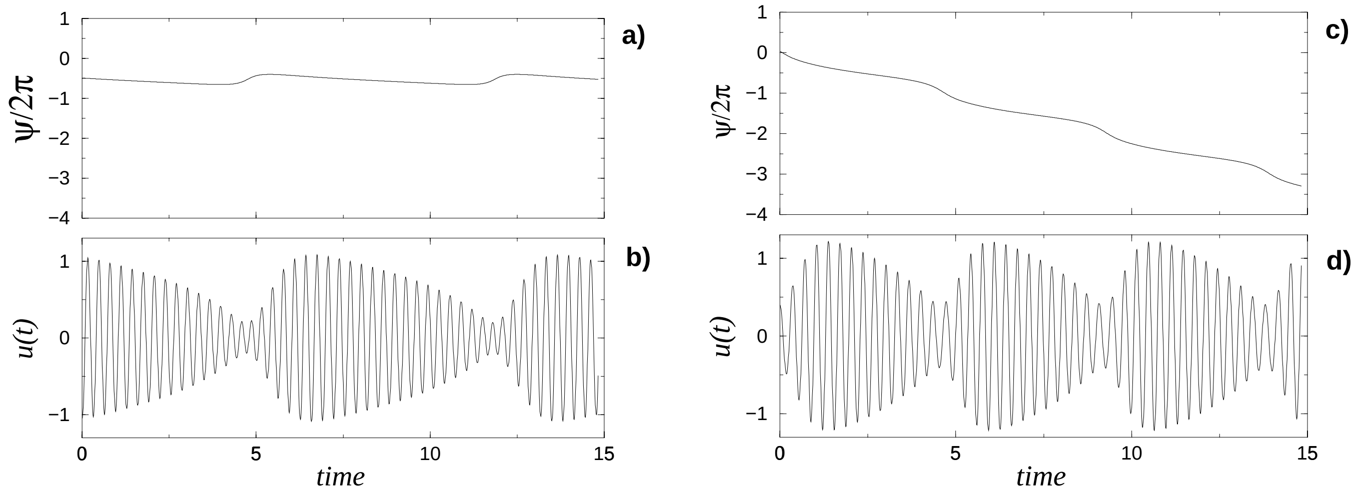

Fig. 4. Oscillations in the forced oscillator (7) at large forcing amplitudes. (a) Inside the synchronization region, but near the transition border, the phase difference is modulated, although bounded; (b) the amplitude is modulated as well. (c) After the transition, the phase difference ψ rotates nonuniformly, but without epochs of "nearly-synchronous" behavior [cf. Fig. 2(c)]. (d) The amplitude modulation is relatively strong [cf. Fig. 2(d)].

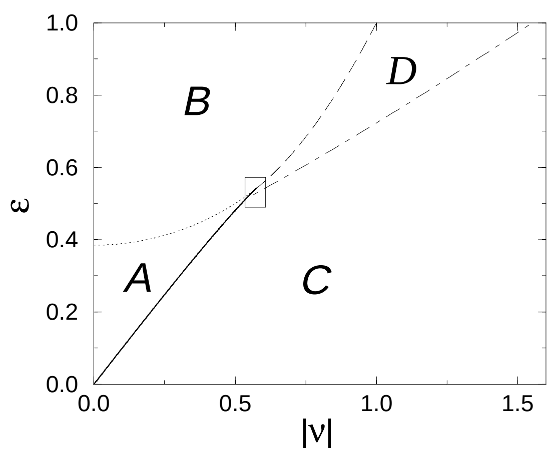

Fig. 5. The bifurcation diagram for Eq. (7) in dependence on the frequency mismatch ν and the forcing amplitude ε. In regions A, B there is a stable fixed point, corresponding to a stable synchronized state (in A there is additionally a pair of unstable fixed points). In regions C, D there is a stable limit cycle. The difference is in the form and amplitude of the cycle: in D it does not envelope the origin, while in C a point on the cycle rotates around the origin. A transition from A to C occurs via saddle-node bifurcation (bold line), it is illustrated in Fig. 1. A transition $B \to D \to C$ is illustrated in Fig. 3; the Andronov–Hopf bifurcation is shown by a dashed line; the transition from frequency-locked to nonsynchronous state is shown by dashed–dotted line. In the region shown by a box, complex bifurcations around the Takens–Bogdanov point occur, see [Holmes & Rand, 1978; Argyris *et al.*, 1994] for details.

3. Noisy Oscillators

In this section we describe briefly the influence of noise on phase synchronization [Stratonovich, 1958, 1963]. The most simple way to model a noisy environment is to add a noisy term to Eq. (2), or, for the simplest possible situation, add to Eq. (4):

$$\frac{d\psi}{dt} = \omega_0 - \omega + \varepsilon q(\psi) + \xi(t)\,. \qquad (8)$$

The dynamics of the phase can be treated as the dynamics of an overdamped particle in a potential

$$V(\phi) = (\omega - \omega_0)\phi - \varepsilon \int^{\phi} q(x)dx\,.$$

The average slope of the potential is determined by the mismatch of frequencies of the autonomous oscillator and external force; the depth of the minima (if they exist) is determined by the amplitude of the forcing, Fig. 6. Without noise, the particle would either rest in a minimum, or slide downwards along the potential, if there are no local minima; this corresponds to synchronous and nonsynchronous states, respectively.

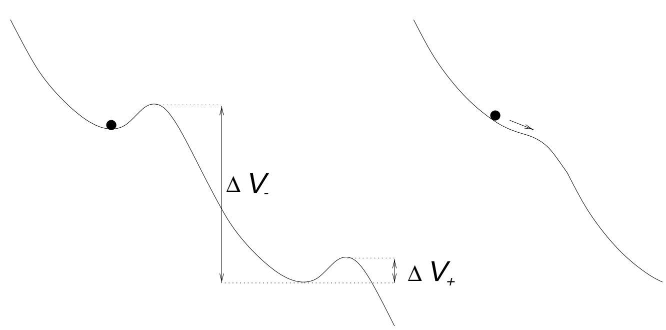

Fig. 6. Phase as a particle in an inclined potential, inside and outside of the synchronization region.

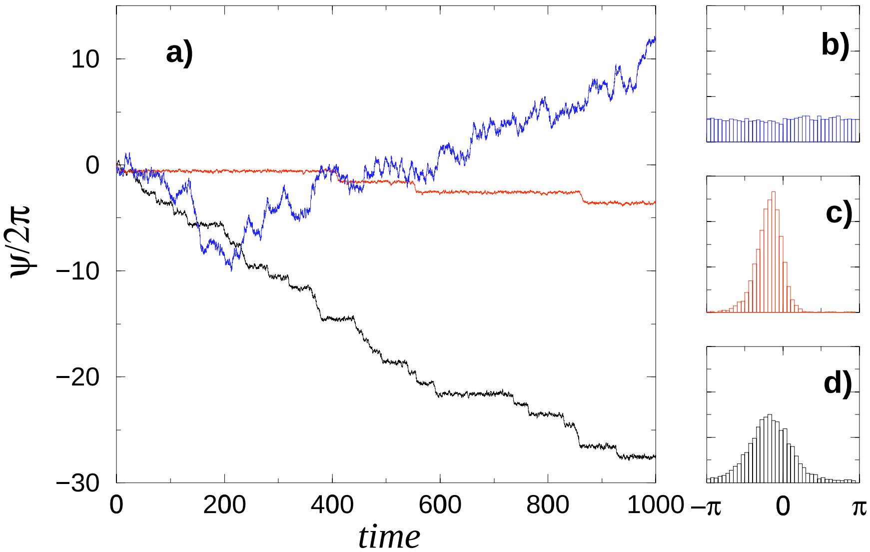

Fig. 7. (a) Fluctuation of the phase difference in a noisy oscillator. Without forcing, the behavior of the ψ is diffusive: It performs a motion that reminds a random walk (blue curve); the distribution of the $\psi \bmod 2\pi$ is shown in (b), it is practically uniform. External forcing with nonzero detuning suppresses the diffusion, the phase of the oscillator is nearly locked (red curve), but sometimes phase slips occur; the respective distribution (c) becomes rather narrow and unimodal. Stronger noise (black curve) causes more phase slips, so that there are only rather short epochs where ψ oscillates around a constant level; the distribution of the $\psi \bmod 2\pi$ remains nevertheless unimodal (d).

Suppose first that the noise is small and bounded, then its influence results in fluctuations of the particle around a stable equilibrium, i.e. in fluctuations of the phase difference around some constant value. We thus have a situation of phase locking in the sense of relation (5); here the observed frequency coincides with that of the external force.

Contrary to this, if the noise is unbounded (e.g. Gaussian), there is always a probability for the particle to overcome a potential barrier ΔV and to hop in a neighboring minimum of the potential. The time series looks like a sequence of these phase slips (see Fig. 7) and relation (5) does not hold. Nevertheless, at least for small noise, the phase synchronization is definitely detectable, although it is not perfect: between slips we observe epochs of phase locking. Averaged locally over such an epoch, the frequency of the oscillator coincides with that of the external force. The observed frequency that is computed via averaging over a large period of time differs from that of the external force, but this difference is small if the slips are rare.

Phase locking in noisy systems can be also understood in a statistical sense, as an existence of a preferred value of the phase difference $\psi \bmod 2\pi$. Indeed, the particle spends most of the time around a position of stable equilibrium, then rather quickly it jumps to a neighboring equilibrium, where the phase difference differs by a multiple of 2π. This can be reflected by distribution of $\psi \bmod 2\pi$: A non-synchronous state would have a broad distribution, whereas synchronization would correspond to a unimodal distribution (Fig. 7).

The synchronization transition in noisy oscillators appears as a continuous decrease of characteristic time intervals between slips, and is smeared: We cannot unambiguously determine the border of this transition.

4. Chaotic Oscillators

For a periodic oscillator the phase was introduced in Eq. (1) as a variable corresponding to the shift along the limit cycle, and, hence, to the zero Lyapunov exponent. Any autonomous continuous-time dynamical system with chaotic behavior possesses one zero Lyapunov exponent that corresponds to shifts along the flow, therefore we expect that phase can be introduced for this case as well.

Suppose we define a Poincaré secant surface for our chaotic system. Then, for each piece of a trajectory between two cross-sections with this surface we define the phase as a linear function of time, so that the phase increment is 2π at each rotation:

$$\phi_P(t) = 2\pi \frac{t - t_n}{t_{n+1} - t_n} + 2\pi n\,, \quad t_n \le t < t_{n+1}\,. \tag{9}$$

Here t_n is the time of the nth crossing of the secant surface.

Obviously, this definition is ambiguous, because it depends on the choice of the Poincaré surface. Nevertheless, defined in this way, the phase has a physically important property: Its perturbations neither grow nor decay in time, so it does correspond to the direction with the zero Lyapunov exponent in the phase space. Note that for periodic oscillations corresponding to a fixed point of the Poincaré map, this definition gives the correct phase satisfying Eq. (1).

It is important to emphasize that the phase of a chaotic oscillator can be also introduced by means of two other techniques:

(1) Quite often we can find a projection of the strange attractor onto a plane such that the trajectory always revolves around some point that can be taken as the origin. In this case the instantaneous phase $\phi(t)$ can be identified with the angle between some chosen direction in this plane and a vector drawn from the origin to a corresponding point of the trajectory.

(2) Taking any oscillatory observable $s(t)$ of a chaotic system, one can construct the so-called *analytic signal* [Gabor, 1946]
$$\zeta(t) = s(t) + is_H(t) = A(t)e^{i\phi(t)}\,, \qquad (10)$$
where $s_H(t)$ is the Hilbert transform (HT) of $s(t)$ and unambiguously obtain the instantaneous phase $\phi(t)$; introduction to the method can be found in [Panter, 1965], practical implementation is discussed in [Rosenblum & Kurths, 1998; Rosenblum *et al.*, 2000]. It is important to mention that these three different approaches to phase determination give practically coinciding results, at least if the system is a "good" one [Pikovsky *et al.*, 1997b].

In contrast to the case of periodic oscillations, the growth of the phase of a chaotic system cannot generally be expected to be uniform. Indeed, the Poincaré return times (they can be taken as an instantaneous period) depend usually on the coordinate of the intersection with the Poincaré surface, i.e. on the irregular amplitude. This dependence can be considered as an influence of some effective "noise", although this irregularity has of course purely deterministic origin. Thus, the synchronization phenomena for a chaotic system are similar to those in noisy periodic oscillations [Rosenblum *et al.*, 1996; Pikovsky *et al.*, 1997b].

We illustrate the effect of phase synchronization in chaotic systems with two characteristic examples (for other examples see [Anishchenko *et al.*, 1992; Rosenblum *et al.*, 1997]). First, we consider the periodically forced Rössler system
$$\begin{aligned}\dot{x} &= -y - z + \varepsilon\cos\omega t\,,\\ \dot{y} &= x + 0.15y\,,\\ \dot{z} &= 0.4 + z(x - 8.5)\,.\end{aligned} \qquad (11)$$
For the chosen parameter values the structure of the strange attractor is rather simple, and the phase can be easily computed by means of any of the above techniques. As a result we find, that the synchronization properties of this system are similar to those of a periodic oscillator subject to a bounded noise. Indeed, there exist a certain range of the frequency of external force ω such that the phase of the Rössler system is locked to the phase of the external force: The phase difference fluctuates in a random manner around some constant level, and the condition (5) is fulfilled. No phase slips are observed; they appear and become more and more frequent when ω approaches the border of the synchronization region. Synchronization here can be also understood in terms of instantaneous periods: The times T_i of return to a Poincaré secant surface are not constant, but in the synchronous state the average return time becomes equal to the period of the external drive, $\langle T_i\rangle = 2\pi/\omega$ (Fig. 8).

We emphasize that the phase synchronization of a chaotic system appears exactly as the relation (5) between phases of the oscillator and external force; the amplitudes remain chaotic and are practically not correlated with the amplitude of the force.

As the second example we take the well-known Lorenz system
$$\begin{aligned}\dot{x} &= 10(y - x)\,,\\ \dot{y} &= 28x - y - xz\,,\\ \dot{z} &= -8/3\cdot z + xy + \varepsilon\cos\omega t\end{aligned} \qquad (12)$$
with the additional term describing the periodic forcing. In the traditional representation of the attractor (projection onto the (x, y) plane) trajectory rotates around two centers, and introduction of the phase seems to be impossible. Nevertheless, if we consider the projection of the phase space onto the plane ($u = \sqrt{x^2 + y^2}$, z) [see Fig. 9(a)], then the phase can be easily determined either via Poincaré section, or as the angle in this plane, i.e. $\phi(t) = \arctan[(z - z^*)/(u - u^*)]$, where z^* and

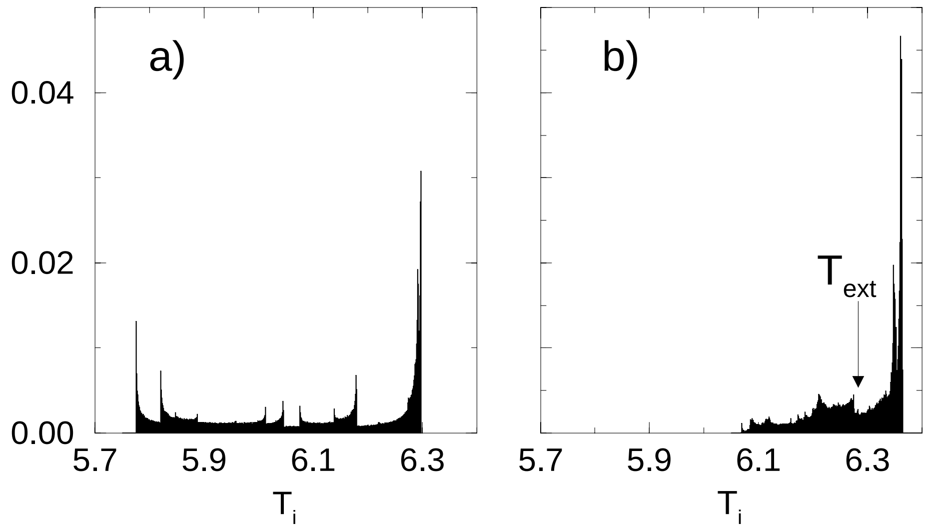

Fig. 8. Distribution of return times for the (a) autonomous and (b) forced Rössler oscillator. For the forced system, the average return time is equal to the period of the driving force.

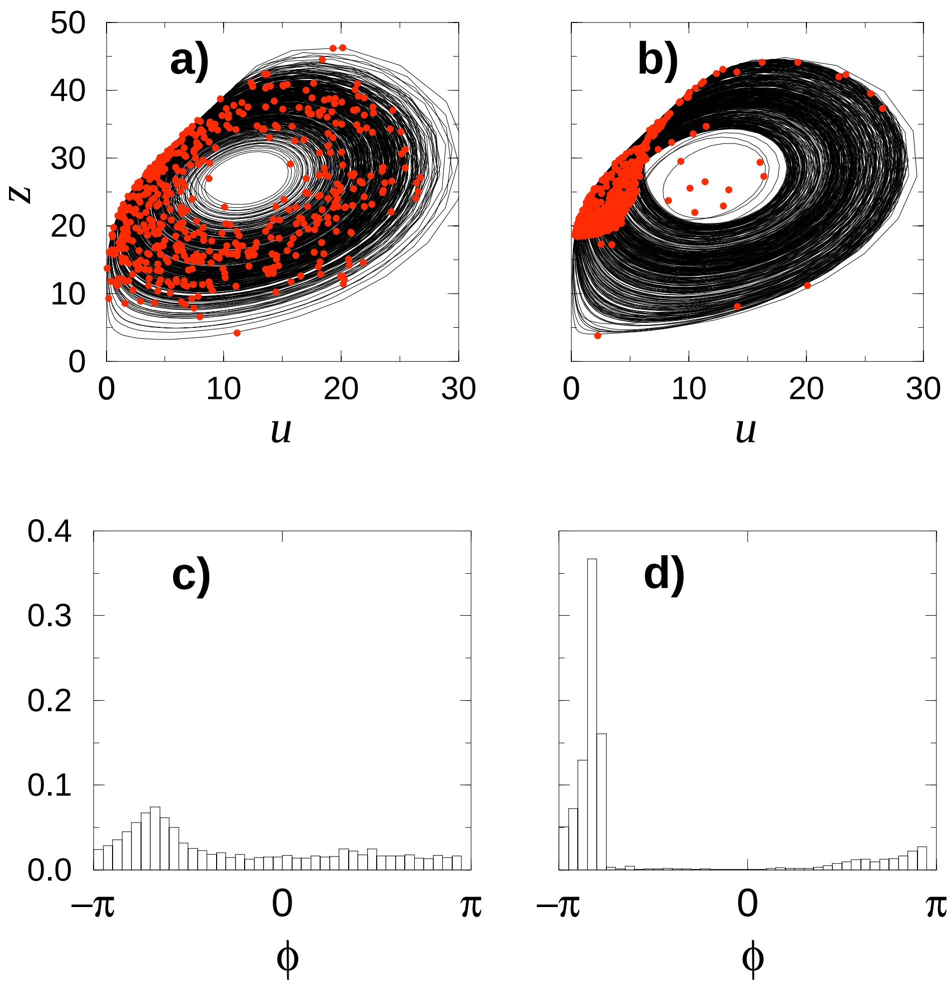

Fig. 9. (a) Attractor of the Lorenz system in coordinates $(u = \sqrt{x^2 + y^2}, z)$. The coordinates of 500 initially closed points are shown by red dots. (c) This illustrates the phase diffusion in the unforced system: The trajectories with initial practically coinciding phases now have phases that are almost uniformly distributed. (b and d) In the presence of forcing, the phase diffusion is suppressed, and the phases of almost all different trajectories follow the phase of the drive.

u^* are the coordinates of the point that is enclosed by the trajectory. Alternatively, the phase can be computed with the help of the Hilbert transform from the oscillatory observable $z(t)$; all the methods giving practically coinciding results.

Consideration of the phase difference shows that synchronization of the Lorenz system can be achieved, but it is not perfect: epochs of the phase locking are always interrupted by phase slips. These slips are persistent, and occur predominantly in one direction, so that the frequency of the oscillator differs slightly from that of the external force, for details see [Zaks *et al.*, 1999].

Here we illustrate synchronization in the Lorenz system in the following way. Let us follow the evolution of trajectories in an ensemble of 500 identical oscillators driven by the same force. The initial conditions for all the oscillators are very close, they differ by less than 10^{-5}. As the systems are chaotic, these initially close trajectories diverge; in Fig. 9 we show the coordinates of all the systems after some evolution time t. In the absence of forcing, the points scatter over the whole attractor [Fig. 9(a)]; in other words, the oscillators with practically the same initial phase have now a value of phase that is almost equally distributed in the interval $[-\pi, \pi]$ [Fig. 9(c)]. External forcing suppresses this phase diffusion, and the phases of the elements of the ensemble concentrate now near the phase of the drive; the distribution becomes rather narrow [Figs. 9(b) and 9(d)]. Such a statistical method of characterization of synchronization becomes especially important in the case when the phase of an oscillator is not well-defined, see [Pikovsky *et al.*, 1997b].

We conclude this section with three important comments:

(1) We have defined the phase of a chaotic system as a coordinate corresponding to the zero Lyapunov exponent. Low-dimensional systems have only one zero exponent, therefore the oscillator can have only one phase, in spite of the fact that sometimes the trajectory rotates around two centers. In this case one has to look for an appropriate projection, although there are no general recipes how to do it.

(2) Another problem arises if there is no distinct center of rotation at all; nevertheless, the existence of zero Lyapunov exponent suggests that the phase should exist as well. In this case, synchronization can be characterized indirectly, without explicit computation of the phase itself [Pikovsky *et al.*, 1997b].

(3) Finally, we would like to mention the works [Pikovsky *et al.*, 1997a; Rosa Jr. *et al.*, 1998; Lee *et al.*, 1998] where the synchronization–desynchronization transition has been studied. This transition can be viewed as a generalization of the saddle-node bifurcation (described in Sec. 2 above) to the case of chaotic attractors. The transition is smeared; the best way to see this is to look on the unstable cycles embedded in the chaotic set (for the synchronization properties of unstable cycles inside chaos see also [Zaks *et al.*, 1999, 2000]). Each of these cycles undergoes the saddle-node bifurcation, and these bifurcations occur at different values of the parameters. The whole transition can be understood as a repeller–attractor collision, where the repeller and the attractor consist of the trajectories that are unstable and stable with respect to the phase, correspondingly. The situation here is similar to the smeared pitchfork bifurcation at the transition to *complete* synchronization of interacting chaotic systems (called symmetry-breaking or blowout bifurcation, see, e.g. [Pikovsky & Grassberger, 1991; Maistrenko *et al.*, 1998]).

5. From Theory to Data Analysis

In this section we discuss how the concept of phase synchronization can be used in order to reveal the presence of interaction between systems from experimental data. We have already summarized that synchronization of weakly coupled oscillators appears as some relation between their phases and frequencies. In the context of data analysis we are going to exploit this fact to tackle the inverse problem: Our goal is to infer the presence of synchronization from data. To this end we have to estimate from the signals the phases and the frequencies, and to look for relations between them.

Generally, we try to access the following problem: Suppose we observe a system with a complex structure that is not known exactly, and measure two time series at its outputs. Our goal is not only to find out whether these signals are dependent or not (this can be done by means of traditional statistical techniques), but to extract additional information on the interaction of some subsystems within the system under study. Naturally, we need some additional knowledge about the observed objects in order to assume that they are self-sustained oscillators having their own rhythms, that may be (or may be not) adjusted due to interaction.

An advantage of this approach is that it allows one to study rather weak interactions between the two oscillatory subsystems. Indeed, the notion of phase synchronization implies only some interdependence between phases, whereas the irregular amplitudes may remain uncorrelated. The irregularity of amplitudes can mask the phase locking so that traditional techniques treating not the phases but the signals themselves may be less sensitive in the detection of the systems' interrelation [Rosenblum *et al.*, 1997, 1998].

The first step in this data analysis is to *estimate* the phases from scalar signals. For this goal we can adapt the above discussed methods of phase determination. Sometimes the signal can be reduced to a sequence of events (spike train). A typical example is a human electrocardiogram (ECG) that can be substituted by a series of R-peaks that appears at time instants t_k. The interval between two R-peaks corresponds to one complete cardiocycle, therefore the phase increase during this time is exactly 2π. Hence, we can assign to the times t_k the values of phase $\phi(t_k) = 2\pi k$, and take it linearly growing for the interval $t_k < t < t_{k+1}$. The determination of phase via marker events in time series can be considered as the analogy to the technique of Poincaré section [see Eq. (9)], although we do not need to assume that the system under study is a dynamical one.

If the signal looks like a sine-wave with slowly varying frequency and amplitude, then its phase can be obtained by means of the *analytic signal concept* based on the Hilbert Transform [Eq. (10)]. We can look at this technique also from another viewpoint: It can be considered as a two-dimensional embedding in coordinates $(s(t),\, s_H(t))$. Note that in these coordinates a harmonic oscillation is represented by a circle for any ω. This circle can be considered as an analog to the phase portrait of the harmonic oscillator. The phase obtained from this portrait increases linearly in time $\phi(t) = \omega t + \phi_0$, as we expect it for this system. Note, however, that the often used coordinates $(s(t),\, \dot{s}(t))$ and delay coordinates $(s(t),\, s(t-\tau))$ generally produce an ellipse; the phase obtained as an angle from such plots demonstrates periodic deviation from the linear growth (i.e. $[\phi(t)-\omega t]$ oscillates periodically) as an artifact of embedding.[3]

The next step in the data processing is to analyze the behavior of the estimated phase difference $\psi = \phi_1 - \phi_2$, or, in a more general case, $\psi_{n,m} = n\phi_1 - m\phi_2$ (for the phases estimated from the signals we use the same notation as for correct phases that have been introduced in Sec. 2). Sometimes synchronization can be detected in a straightforward way: by plotting $\psi_{n,m}$ versus time and looking for horizontal plateaus in this presentation.

To illustrate this, we describe the results of experiments on posture control in humans [Rosenblum *et al.*, 1998]. During these tests a subject is asked to stay quietly on a special rigid force plate equipped with four tensoelectric transducers. The output of the setup provides current coordinates $(x,\, y)$ of the center of pressure under the feet of the standing subject. These bivariate data are called stabilograms; they are known to contain rich information on the state of the central nervous system [Gurfinkel *et al.*, 1965; Cernacek, 1980; Furman, 1994; Lipp & Longridge, 1994]. Every subject was asked to perform three tests of quiet upright standing (3 min) with

(a) eyes opened and stationary visual surrounding (EO);
(b) eyes closed (EC);
(c) eyes opened and additional video-feedback (AF).

132 bivariate records obtained from 3 groups of subjects (17 healthy persons, 11 subjects with an organic pathology and 17 subjects with a phsychogenic pathology) were analyzed by means of cross-spectra and generalized mutual information. It is important that the interrelation between body sway in anterior–posterior and lateral directions was found in pathological cases only. Another observation is that stabilograms can be qualitatively rated into two groups: noisy and oscillatory patterns. The latter appears considerably less frequently — only some few per cent of the records can be identified as oscillatory — and only in the case of pathology.

The appearance of oscillatory regimes in stabilograms suggests excitation of self-sustained oscillations in the control system responsible for the maintenance of the constant upright posture; this system is known to contain several nonlinear

[3]Obviously, to obtain for a harmonic signal a circle in the embedding, one can use the coordinates $(s(t),\, \dot{s}(t)/\omega)$ or delay coordinates with $\tau = \pi/2\omega$, but this requires a prior knowledge of ω and cannot be implemented for a signal with slowly varying frequency.

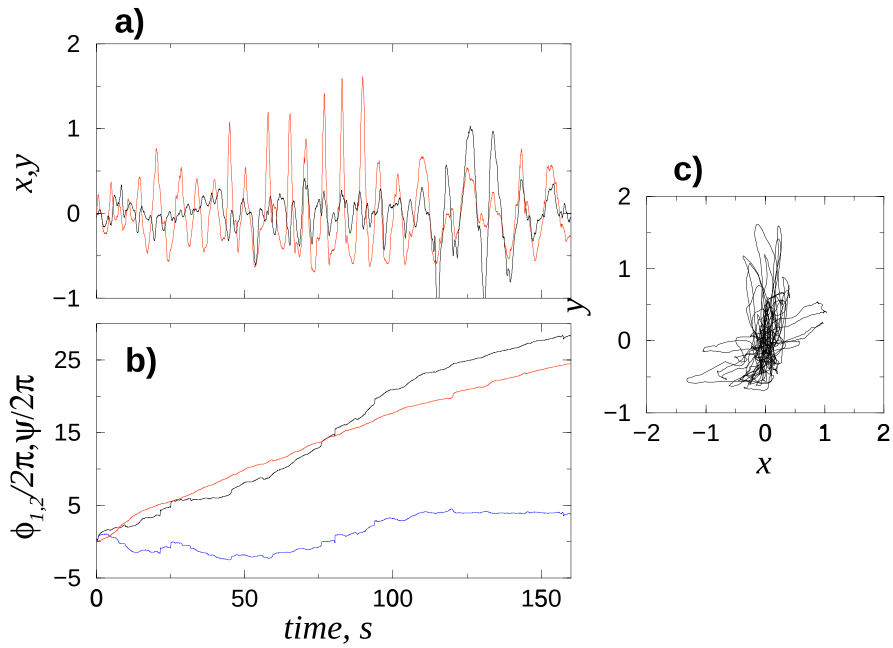

Fig. 10. Stabilogram of an neurological patient. (a) x (black curve) and y (red curve) represent the body sway while quite stance with open eyes in anterior–posterior and lateral directions, respectively. (b) The phases of these signals, and the phase difference are shown by black, red and blue. The transition to a synchronous regime is clearly seen at ≈ 110 sec. (c) The plot of y versus x shows no structure indicating the interrelation between the signals.

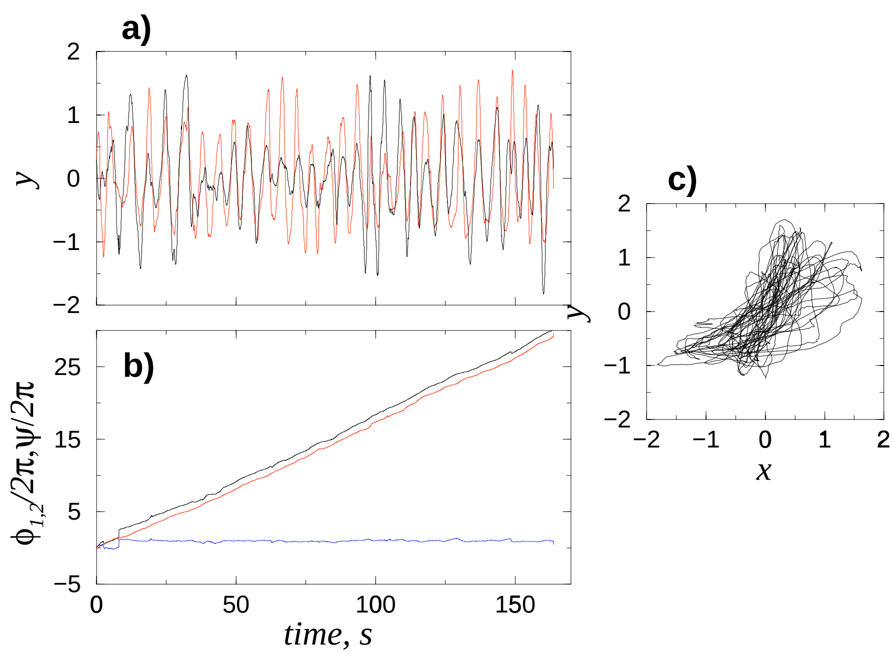

Fig. 11. Stabilogram of the same patient as in Fig. 10 obtained during the test with the eyes closed. All the notations are the same as in Fig. 10. From the phase difference one can see that the body sway in two directions are synchronous within the whole test, although the amplitudes are irregular and essentially different.

feedback loops with time delay. On the other hand, the independence of body sway in two perpendicular directions for all healthy subjects and many cases of pathology suggests that two separate subsystems are involved in the regulation of the upright stance. A plausible hypothesis is that when self-sustained oscillations are excited in both these subsystems, synchronization may take place. To test whether the interdependence of two components of a stabilogram may be due to synchronization, we have performed the analysis of the relative phase.

Here we present the results for one trial (female subject, 39 years old, functional ataxia). We can see that in the EO and EC tests the stabilograms are clearly oscillatory (Figs. 10 and 11). The difference between these two records is that with eyes opened the oscillations in two directions are not synchronous during approximately the first 110s, but are phase locked during the last 50s. In the EC test, the phases of oscillations are perfectly entrained all the time. The behavior is essentially different in the AF test; here no phase locking is observed. We emphasize, that the traditional techniques fail to detect the cross-dependence of these signals because of the nonstationarity and insufficient length of the time series.

This simple method of synchronization analysis proved to be efficient in the investigation of model systems as well as for some experimental data. Nevertheless, quite often the noise level and the nonstationarity in the system are very high, and the synchronization region is rather narrow, so that direct inspection of phase dynamics is hardly successful. In these cases special data processing techniques are required in order to reveal the evidence of a very weak interaction and quantify it, see [Schäfer *et al.*, 1998; Tass *et al.*, 1998; Schäfer *et al.*, 1999; Rosenblum *et al.*, 2000].

6. Conclusions

In this tutorial article we have discussed the main concepts of phase synchronization. This phenomenon lacks a unique definition. For periodical systems synchronization can be understood as phase or frequency locking; in the general case of $n : m$ synchronization these conditions can be written as $|n\phi_1 - m\phi_2 - \delta| <$ const and $n\Omega_1 = m\Omega_2$, where $\Omega_{1,2} = \langle\dot{\phi}_{1,2}\rangle$ and $\phi_{1,2}$ are phases of two interacting oscillators. For noisy and chaotic systems, understanding of synchronization may be ambiguous, and the transition to the synchronous state is always smeared. Sometimes, synchronization in these systems also appears as the phase locking in the above sense, but more often this property is observed as a tendency, or as a temporary event on some finite time intervals only. Here a statistical description of phase difference is required. In a noisy/chaotic case there may be no parameter range where the frequencies of coupled oscillators coincide exactly, but they get closer due to interaction.

Theory of synchronization is a rapidly developing branch of nonlinear science. Finally we mention some ongoing directions. An interesting field is synchronization in large ensembles of oscillators (population of globally coupled systems, chains and lattices), it is related i.e. to behavior in neural ensembles and other biological systems, see [Gerstner, 1995; Ernst *et al.*, 1998] and references there. A popular paradigmatic model in this context is a system of pulse-coupled integrate-and-fire oscillators, see e.g. [Mirollo & Strogatz, 1990]. Synchronization in spatially-distributed systems was considered, e.g. in [Osipov *et al.*, 1997; Goryachev *et al.*, 1998; Chaté *et al.*, 1999]. Other synchronous states in chaotic systems, namely full (identical), generalized and lag synchronization, as well as synchronization transitions are described in [Ditto & Showalter, 1997; Schuster, 1999] and references therein. The extension of the notion of phase synchronization to stochastic systems exhibiting the effects of stochastic and coherence resonance can be found in [Shulgin *et al.*, 1995; Neiman *et al.*, 1999; Han *et al.*, 1999].

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