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Detecting direction of coupling in interacting oscillators

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We propose a method for experimental detection of directionality of weak coupling between two self-sustained oscillators from bivariate data. The technique is applicable to both noisy and chaotic systems that can be nonidentical or even structurally different. We introduce an index that quantifies the asymmetry in coupling.

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Development of nonlinear dynamics essentially contributed to assortment of contemporary time series analysis [1]. Still, the analysis of multivariate data remains a challenge. A problem of particular interest is to assess the interaction between two (sub)systems by means of the analysis of interrelation between two signals at their outputs. An important application is, e.g., the analysis of normal and pathological brain activity from multichannel noninvasive electro- or magnetoencephalography measurements [2–4].

One approach to this problem is based on the idea of dynamical interdependence, or generalized synchronization of unidirectionally coupled systems [3,5,6]. The measures of driver-response relationships obtained by means of this approach are not always reliable for noisy data; they mainly reflect the different degrees of complexity of the two systems, and the interpretation of this information is difficult [7]. Another approach exploits the notion of (phase) synchronization of irregular oscillators [8,9]; it is based on the assumption that the system under study can be modeled by coupled self-sustained systems [2,9,10]. This approach makes use of the well-known fact that weak coupling first affects the phases of the oscillators, not their amplitudes. Hence, in order to reveal and quantify the strength of interaction one has to analyze relation between the phases of the systems. Nevertheless, up to now, there exists no methods to estimate the direction of coupling from such data.

In this contribution we develop a technique that allows us to reveal whether the interaction is bi- or unidirectional and quantify the degree of asymmetry in the coupling. The merits of the method are (a) its ability to detect and quantify rather weak interaction between oscillators, even if it is too weak to induce synchronization, and (b) its applicability to both noise-perturbed and chaotic systems. Thus, we avoid main limitations of the methods based on the notion of generalized synchronization.

Our assumption is that the two observed signals represent two weakly interacting oscillators. Using standard methods [11] one can estimate from these data the time series of phases $\phi_{1,2}(t_k)$, where $t_k=\delta tk$, δt is the sampling interval, $k=1,\ldots,N$. The principal idea, illustrated in Fig. 1(a), is to look whether the phase dynamics of one oscillator is influenced by the phase of the other. To explain the proposed technique we start with a simple model of two coupled phase oscillators,

$$\begin{aligned}\dot{\phi}_1&=\omega_1+q_1(\phi_1)+\varepsilon_1 f_1(\phi_1,\phi_2)+\xi_1(t),\\ \dot{\phi}_2&=\omega_2+q_2(\phi_2)+\varepsilon_2 f_2(\phi_2,\phi_1)+\xi_2(t).\end{aligned} \quad (1)$$

Here ϕ_1,ϕ_2 are phase variables, so that the functions $q_{1,2}, f_{1,2}$ are 2π-periodic in all arguments, and the phase space of the model is the two-dimensional torus; parameters $\omega_{1,2}$ govern the natural frequencies of oscillators (although do not coincide with them for $q_{1,2}\neq 0$). To take into account noisy perturbations (that are always present in natural systems) we include in Eqs. (1) random terms $\xi_{1,2}$. System (1) describes the phase dynamics of weakly coupled noisy limit cycle oscillators, Josephson junctions, and phase locked

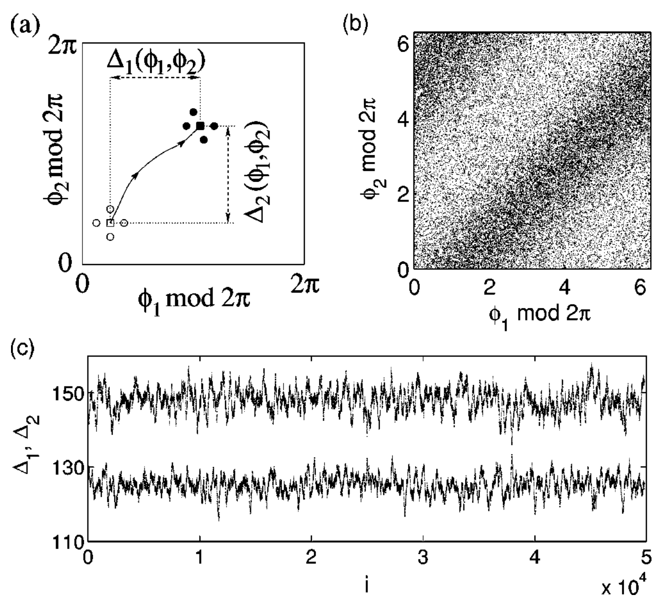

FIG. 1. (a) Evolution of neighboring trajectories on the torus (ϕ_1,ϕ_2) during the time interval τ. The points denoted by open symbols evolve with time to positions shown by closed ones; for clarity, only one trajectory is shown by the arrowed line. In the example shown here (schematically), the increment of ϕ_1, $\Delta_1=\phi_1(t+\tau)-\phi_1(t)$, depends on ϕ_2, hence, the phase of the first oscillator is influenced by the second one. On the contrary, the increment Δ_2 is constant, what indicates that there is no coupling in the direction from 1 to 2. (b) Phases of coupled oscillators [Eq. (1)] for $\varepsilon_2=0.02$, $\tau=40\pi$. (c) The dependence of $\Delta_{1,2}$ (upper and lower curves, respectively) on index i.

loops [12], as well as phase dynamics of weakly coupled continuous-time chaotic systems [8].

For $\omega_1 \approx \omega_2$ and vanishing noise system (1) has a limit cycle solution on the torus, which corresponds to the synchronous state (mode locking). Outside of the synchronization region on the parameter plane (i.e., for large $|\omega_1 - \omega_2|$ or for small $\varepsilon_{1,2}$) the state is quasiperiodic with ergodic dynamics on the torus. With sufficiently large noise one cannot distinguish between these cases. We emphasize that in the synchronous state there is a definite relation between the phases determined by $\varepsilon_{1,2}$ as well as by $\omega_{1,2}$, $q_{1,2}$, and $f_{1,2}$. Thus, in case of perfect synchrony, we are not able to separate the effect of interaction from the internal dynamics of autonomous systems. In order to obtain the information on the direction of coupling we need to observe deviations from the synchrony, either due to noise or due to onset of quasiperiodic dynamics outside the synchronization region. Below we give examples of both situations.

Our goal is to estimate the ratio of the coupling terms from the time series of the phases $\phi_{1,2}(t_k)$ only [the phases are unwrapped, i.e., not reduced to the interval $[0, 2\pi)$]. First, we compute for each time point the increments $\Delta_{1,2}(k) = \phi_{1,2}(t_k + \tau) - \phi_{1,2}(t_k)$; the choice of the constant τ is not very important, as shown below. These increments can be considered as generated by some unknown two-dimensional noisy map $\Delta_{1,2}(k) = \mathcal{F}_{1,2}[\phi_{1,2}(k), \phi_{2,1}(k)] + \eta_{1,2}(k)$. Next, we fit (in the least mean square sense) the dependencies of Δ on ϕ_1 and ϕ_2 using a finite Fourier series as the probe function:

$$F_{1,2} = \sum_{m,l} A_{m,l} e^{im\phi_1 + il\phi_2};$$

in the following computations we take the terms with $|l| \leq 3$ for $m=0$, $|m| \leq 3$ for $l=0$, and $|m| = |l| = 1$. The smooth functions $F_{1,2}$ are estimates of the deterministic parts $\mathcal{F}_{1,2}$ of the above map. A similar procedure was used for noise reduction in discrete [13] dynamical systems and (with $\tau \to 0$) for extracting model equation from experimental noisy data [14]. The results of fitting are used to quantify the cross-dependencies of phase dynamics of two systems by means of the coefficients $c_{1,2}$ defined as

$$c_{1,2}^2 = \int\int_0^{2\pi} \left(\frac{\partial F_{1,2}}{\partial \phi_{2,1}} \right)^2 d\phi_1 d\phi_2 . \qquad (2)$$

Finally, we calculate the *directionality index* as

$$d^{(1,2)} = \frac{c_2 - c_1}{c_1 + c_2}. \qquad (3)$$

Normalized in this way, the index varies from 1 in the case of unidirectional coupling $(1 \to 2)$ to -1 in the opposite case $(2 \to 1)$; vanishing index $d^{(1,2)} = 0$ corresponds to symmetric bidirectional coupling [15].

We illustrate the described technique with the simplest example, where we simulate the coupled phase oscillators system (1) with $\omega_{1,2} = 1 \pm 0.1$, $q_{1,2} = 0$, $f_{1,2} = \sin(\phi_{2,1} - \phi_{1,2})$, and Gaussian δ-correlated noisy perturbations $\xi_{1,2}$,

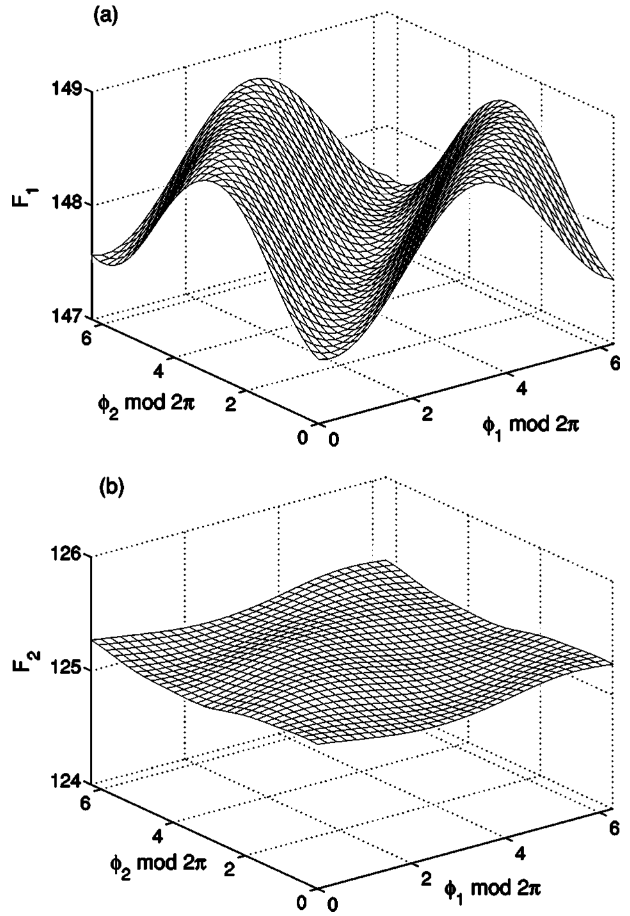

FIG. 2. Estimated functions $F_1(\phi_1, \phi_2)$ (a) and $F_2(\phi_1, \phi_2)$ (b) for the same parameters as used in Figs. 1(b) and 1(c) clearly reveal asymmetry in coupling. Increment of ϕ_1 strongly depends on both $\phi_{1,2}$ having the form $F_1 \approx F_1(\phi_2 - \phi_1)$, as it should be for the model considered. For the second phase, the dependence is weaker. Note that fitting plays the role of averaging (noise reduction) and provides smooth functions $F_{1,2}$ starting from the irregular $\Delta_{1,2}(i)$ [cf. Fig. 1(c)]

$\langle \xi_{1,2}(t) \xi_{1,2}(t') \rangle = \delta_{1,2} \cdot 2D_{1,2} \delta(t - t')$, with $D_{1,2} = 0.2$ [16]. One coupling coefficient was fixed, $\varepsilon_1 = 0.1$, and the other one, ε_2, was varied from 0 to 0.15. The case $\varepsilon_2 = 0.02$ is illustrated in Figs. 1(b) and 1(c); the corresponding approximating functions $F_{1,2}$ are shown in Fig. 2. The results of the calculation of the directionality index are summarized in Fig. 3(a). Because of strong noise in the system, rather large statistics is required. In Fig. 3(b) we report the effect of the time constant τ on the directionality index. One can see that this dependence in a broad range (from $\approx$ 0.5 to $\approx$ 50 periods of oscillation) is very weak.

As the next example, we consider noise-free oscillators, $\varepsilon_1 = 0.05$, $D_{1,2} = 0$ with $q_{1,2} = b\cos(\phi_{1,2})$, and the other parameters the same as above. For these parameters, the coupled system performs a quasiperiodic motion. As is seen from Fig. 4, the information on the asymmetry in coupling is perfectly recovered.

For further illustration we take two coupled van der Pol oscillators:

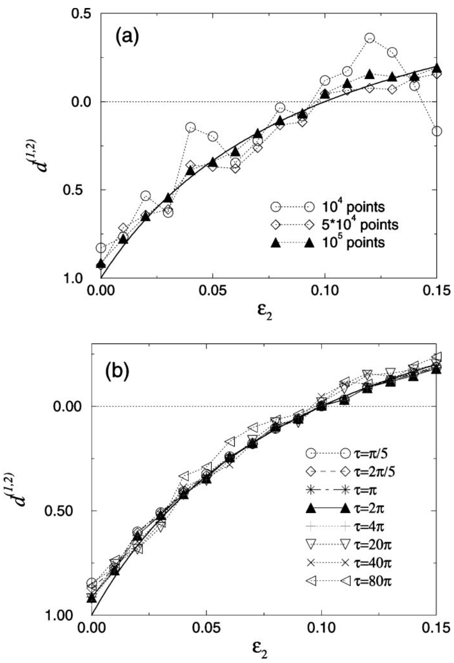

FIG. 3. Dependence of the index $d^{(1,2)}$ on ε_2 for fixed ε_1 for coupled noisy phase oscillators (1). $d^{(1,2)}$ should be zero if the driving is symmetric, and -1 if it is unidirectional (from 2 to 1). The dependence on the length of the time series is shown in (a), whereas the influence of the parameter τ is illustrated in (b). Solid line shows the dependence $y=(\varepsilon_2-\varepsilon_1)/(\varepsilon_2+\varepsilon_1)$. One can see that for sufficiently large data sets the ratio of c_1/c_2 correctly provides the ratio of $\varepsilon_1/\varepsilon_2$.

$$\ddot{x}_{1,2}-0.2(1-x_{1,2}^2)\dot{x}_{1,2}+\omega_{1,2}^2x_{1,2}=\varepsilon_{1,2}(x_{2,1}-x_{1,2})+\xi_{1,2}, \tag{4}$$

where $\omega_{1,2}=1\pm0.02$. The application of the method to this case is essentially the same; the only difference is that first the phases of the systems are estimated from respective signals [11]. For $\varepsilon_{1,2}$ varying in the range from 0 to ≈ 0.05 and noise level varying from $D\approx0.08$ to $D\approx0.12$ we clearly detect the direction of coupling for this model, both for identical and different intensities of noise terms.

The next example shows that the method also works in the case of two structurally different systems, namely, the Rössler and the van der Pol oscillators:

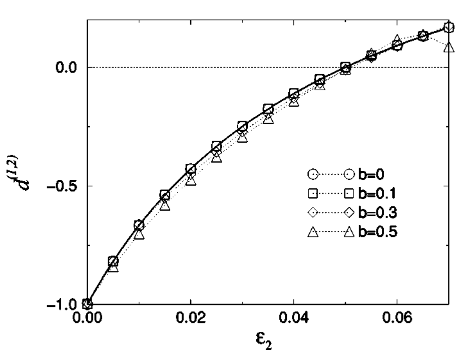

FIG. 4. Directionality index for coupled noise-free phase oscillators (1) in the quasiperiodic regime for different values of b. Solid line shows the dependence $y=(\varepsilon_2-\varepsilon_1)/(\varepsilon_2+\varepsilon_1)$.

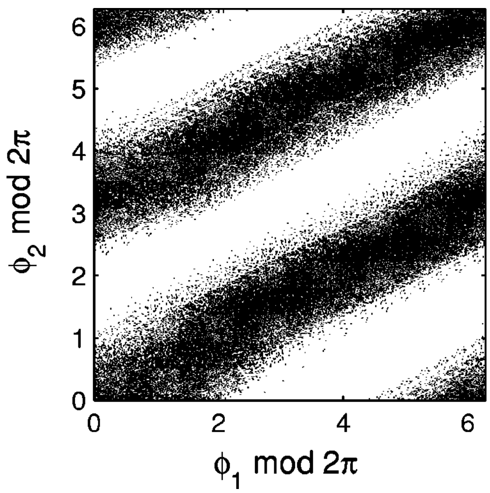

FIG. 5. Phases of two 1:2 synchronized Rössler oscillators (6). The noise in the system is bounded and weak, and therefore the points form a stripe so that the global fitting is not effective. Quasiperiodic states allows better estimation, see text.

$$\begin{aligned}
\dot{x}&=-y-z,\\
\dot{y}&=x+0.15y+\varepsilon_1 u,\\
\dot{z}&=0.2+z(x-10),\\
\ddot{u}-0.1(1-u^2)\dot{u}+\omega_0^2u&=\varepsilon_2 y.
\end{aligned} \tag{5}$$

Here the unidirectional coupling with $\varepsilon_1=0.05$, $\varepsilon_2=0$ or $\varepsilon_1=0$, $\varepsilon_2=0.05$ gives $d^{(1,2)}=-1.0$ and $d^{(1,2)}=0.95$, respectively; $\omega_0=0.98$; the bidirectional coupling, $\varepsilon_1=0.05$, $\varepsilon_2=0.003$, yields $d^{(1,2)}=-0.02$. Note that because the systems are intrinsically different, the asymmetry here is not determined directly by the ratio of ε_1 and ε_2.

We emphasize that the method requires that the trajectory fills the surface of the (ϕ_1,ϕ_2) torus. Indeed, only in this case we can fit the dependencies $\Delta_{1,2}=\mathcal{F}_{1,2}(\phi_{1,2},\phi_{2,1})$ using ϕ_1,ϕ_2 as independent variables. This requirement is fulfilled either if the oscillators are in the quasiperiodic state, or if the noise in the system is sufficiently strong. If the oscillators are noise-free, then in case of synchronization the trajectory on (ϕ_1,ϕ_2) is one line, so ϕ_1 and ϕ_2 are not independent and the functions of two variables $\mathcal{F}_{1,2}$ cannot be estimated.

In the case of weak (or bounded) noise, when the trajectories form smeared stripes (cf. Fig. 5), the above approach must be slightly modified: one should perform local fits only for nonempty regions on the torus (ϕ_1,ϕ_2). The feature of the method not to work for strictly synchronous regimes does not essentially limit practical applications. Indeed, in case of physical experiments, it is usually possible to adjust some parameters and thus to tune the system out of the synchronization region, i.e., to a quasiperiodic state. In biological experiments the system cannot usually be controlled, but the noise inherent to such systems helps. Besides, typically the synchronous regimes are not so probable as the quasiperiodic ones.

Finally, we briefly discuss the case when two systems are close to the state of $n:m$ synchronization. We consider coupled Rössler systems

$$\dot{x}_{1,2}=\omega_{1,2}(-y_{1,2}-z_{1,2})+\varepsilon_{1,2}(x_{2,1}-x_{1,2}),$$
$$\dot{y}_{1,2}=\omega_{1,2}(x_{1,2}+0.15y_{1,2}), \qquad (6)$$
$$\dot{z}_{1,2}=\omega_{1,2}[0.2+z_{1,2}(x_{1,2}-10)],$$

with $\omega_1=1$ and $\omega_2\approx\omega_1/2$. First, we describe the case when the systems are 1:2 locked (see Fig. 5). Generally, in this case the method based on a global approximation fails. Nevertheless, as the stripes here are rather broad and there are two wrappings of the torus in one direction instead of one wrapping for 1:1 locking, we at least correctly estimate the direction of coupling. This is a rough estimate, since only the sign of the coefficient is correct: $d^{(1,2)}=-0.15$ for $\varepsilon_1=0.02$, $\varepsilon_2=0$, and $d^{(1,2)}=0.18$ for $\varepsilon_1=0$, $\varepsilon_2=0.2$. Synchronous states with larger n,m are more favorable, because many revolutions almost cover the surface of the torus. Indeed, for large m,n the motion resembles a quasiperiodic one, where the method is mostly effective. So, the case when the frequency mismatch in Eq. (6) is large enough, and the coupling is not too strong to cause synchronization, admits much better estimation of the directionality index: $d^{(1,2)}=-0.94$ for $e_1=0.05$, $e_2=0$, and $d^{(1,2)}=0.89$ for $e_1=0$, $e_2=0.05$.

In summary, we have proposed a technique of data analysis that provides information on the directionality of coupling between two self-sustained oscillatory systems. The technique is applicable to nonidentical, or even structurally different, oscillators, noisy or chaotic; it can be extended to the case of more than two interacting units. As many natural phenomena can be successfully modeled by coupled self-sustained oscillators, we foresee many practical applications of the technique. Good examples are synchronization of electrochemical and salt-water oscillators [17,18] and mode-locking in lasers [19]. Indeed, in these cases synchronization is observed, whereas the coupling between the (sub)systems cannot be determined and controlled directly. With our method the *a posteriori* estimation of coupling direction becomes feasible. The proposed method can be widely used in biomedical studies (e.g., of cardiorespiratory interaction and brain activity) where the parameters of systems typically cannot be assessed and only measurements under free-running conditions are possible. As one of the directions for further study we outline an optimization of the fitting procedure. As another direction we mention the detailed comparison of the proposed technique with statistical methods based on the computation of mutual predictability or transfer entropy [20].

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